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Non-Abelian Hodge Theory and Moduli Spaces of Higgs Bundles

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Abstract

These notes contain the material presented at one of the mini-courses of the Workshop on Character Varieties and Higgs bundles held in Liberia, Guanacaste, Costa Rica, in August 2025. We sketch the main arguments and basic geometric concepts underlying the "non-abelian Hodge correspondence" relating the moduli space of Higgs bundles over a Riemann surface X with the character variety parametrizing representations of the fundamental group of X. We start by giving a "crash course" on vector bundles and connections on Riemann surfaces. Then, we introduce the classification problem of vector bundles, in order to motivate the natural appearance of moduli spaces, and explain how these moduli spaces can also be constructed in terms of connections. In particular, we state the theorem of Narasimhan-Seshadri as a first instance of non-abelian Hodge theory. Finally, we state the Corlette-Donaldson theorem and the Hitchin-Simpson theorem, which are the main results of non-abelian Hodge theory, and provide an interpretation of these results in terms of moduli spaces.

Contents

1 A primer on vector bundles and connections

1.1 Vector bundles in different categories

Let X be a compact Riemann surface. We can trivialize X by giving a complex atlas: namely, we cover X by a family $\mathfrak{U}$ of open subsets $U \subset X$ with a homeomorphism $\psi_U : U \to D_U \subset \mathbb{C}$ with some disk D_U in $\mathbb{C}$, for each $U \in \mathfrak{U}$, in such a way that the coordinate change functions $\psi_{UV} = \psi_V \circ \psi_U^{-1} : D_U \cap D_V \to D_U \cap D_V$ are holomorphic.

Definition 1. A ***vector bundle*** E of rank r over X is given by glueing spaces of the form $E_U = U \times \mathbb{C}^r$ using a set of continuous transition functions $\{g_{UV} : U, V \in \mathfrak{U}, U \cap V \neq \varnothing\}$, with

$$g_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$$

that satisfy the (1-)***cocycle condition***

$$g_{UV} g_{VW} = g_{UW}.$$

A vector bundle is ***smooth***, ***holomorphic*** or a ***local system*** if the transition functions g_{UV} are respectively smooth, holomorphic or locally constant.

Remark 2. Two vector bundles E and E' are isomorphic if and only if their corresponding 1-cocycles (g_{UV}) and (g'_{UV}) are cohomologous, meaning that there exists some family $\{f_U : U \to \mathrm{GL}_r(\mathbb{C}) : U \in \mathfrak{U}\}$ (i.e. a 0-coboundary) such that

$$g'_{UV} = f_U g_{UV} f_V^{-1}.$$

The action of such 0-coboundaries on the 1-cocycles determines a groupoid, which naturally classifies vector bundles. The set of isomorphism classes of this groupoid can be understood as a "nonabelian sheaf cohomology set" $H^1(X, \mathrm{GL}_r(\mathbb{C})_X)$. Here, $\mathrm{GL}_r(\mathbb{C})_X$ is the sheaf of local functions from X to $\mathrm{GL}_r(\mathbb{C})$; these functions are considered to be smooth, holomorphic or locally constant depending on whether we are considering smooth bundles, holomorphic bundles or local systems.

Remark 3. Any compact Riemann surface is biholomorphic to the analytification C^{an} a smooth complex projective curve C. If we let $\mathfrak{U}$ correspond a cover by Zariski open subsets of C, we can also consider ***algebraic*** vector bundles, determined by the condition that the transition functions g_{UV} are regular maps into GL_r (regarded as a smooth algebraic variety over $\mathbb{C}$). Serre's GAGA theorem [6] implies that the category of algebraic vector bundles over C is equivalent to the category of holomorphic vector bundles over $X = C^{\mathrm{an}}$.

Remark 4. Consider a local system E over X determined by a set of locally constant transition functions $\{g_{UV}\}$. Given any point $x_0 \in X$ we can define the ***monodromy representation*** $\rho_E : \pi_1(X, x_0) \to \mathrm{GL}_r(\mathbb{C})$ constructed as follows. If σ is a loop on X based at x_0, then we can partition the unit interval $t_0 = 0 < t_1 < t_2 < \cdots < t_N = 1$ in such a way that, for every $i = 1, \ldots, N$ there exists some $U_i \in \mathfrak{U}$ such that $\sigma([t_i, t_{i+1}]) \subset U_i$. If we call $x_i = \sigma(t_i)$ and $g_{ij} = g_{U_i U_j}$, we can define

$$\rho_E(\sigma) = g_{12}(x_1) g_{23}(x_2) \ldots g_{N1}(x_N).$$

Exercise 1. Verify the following statements about the remark above.

1. *There exists such a partition of the unit interval. (Hint: Use "Lebesgue's Number Lemma").*
2. *The map $\rho_E(\sigma)$ does not depend on the choice of of "Lebesgue partition".*
3. *The map $\rho_E(\sigma)$ does not depend on the chosen representative of the homotopy class $[\sigma]$.*

1.2 Connections and curvature

We denote by Ω^1_X the (smooth) cotangent bundle of X, and by Ω^k_X its k-th exterior powers. The space of sections $\Omega^k(U)$ of Ω^k_X at an open subset $U \subset X$ is by definition the space of (smooth) differential k-forms on X. Recall that we have exterior differentiation $\Omega^k(U) \to \Omega^{k+1}(U)$. Connections provide a way to generalize exterior differentiation to differential forms with coefficients on a vector bundle. If E is a smooth vector bundle, and $U \subset X$ is an open subset, we denote by $\Gamma(U, E) = \Omega^0(U, E)$ the set of sections of E on U and by $\Omega^k(U, E)$ the sections of $E \otimes \Omega^k_X$ on U.

Definition 5. Let E be a smooth vector bundle on X. A ***connection*** D on E is a $\mathbb{C}$-linear operator

$$D : \Omega^0(X, E) \to \Omega^1(X, E)$$

such that

$$D(fs) = s df + f Ds,$$

for f a smooth function on X and s a section of E on X.

Remark 6. Note that the existence of smooth partitions of unity implies that D can actually be regarded as a map of sheaves of sections

$$D : E \to E \otimes \Omega^1_X.$$

Remark 7. If $U \in \mathfrak{U}$ is an open subset of X in the complex atlas, then the space of sections of E on U is a free $C^\infty(U)$-module of rank r. A basis $\{e_1, \ldots, e_r\}$ of this module is called a ***frame*** of E on U. If D is a connection on E, for each e_i of the frame, the connection acts as

$$De_i = \sum_j e_j A^j_i,$$

for some 1-form $A^j_i \in \Omega^1(U)$. In matrix notation, writing $e = (e_1 \cdots e_r)$ and $A = (A^j_i)$ as a square matrix, we obtain

$$De = eA.$$

Any section of E on U can be written as $s = \sum_i s^i e_i$, for $s^i \in C^\infty(U)$. Therefore, we can write

$$Ds = \sum_i (ds^i e_i + s^i De_i) = \sum_i (ds^i e_i + s^i e_j A^i_j) = (d+A)s$$

The matrix A is called the ***connection*** 1***-form*** of D on U.

The exterior differential d is well known to satisfy the condition d^2. This is however not true in general for connections, which gives rise to the notion of curvature. More precisely, there is a unique way of extending the map $D : \Gamma(X, E) \to \Omega^1(X, E)$ to a map $D : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ in such a way that

$$D(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k \omega \otimes D\alpha,$$

and

$$D(\alpha \wedge \omega) = D\alpha \wedge \omega + (-1)^k \alpha \otimes D\omega,$$

for $\omega \in \Omega^p(X)$ and $\alpha \in \Omega^{k-p}(X, E)$.

Definition 8. Let D be a connection on a smooth vector bundle E. We define the ***curvature*** of D as the operator

$$D^2 : \Gamma(X, E) \to \Gamma(X, E) \otimes \Omega^2(X).$$

Remark 9. The curvature D^2 is a C^∞-linear map since

$$D^2(fs) = D(sdf + fDs) = Ds \wedge df + df \wedge Ds + f \wedge D^2 s = fD^2 s,$$

for $s \in \Gamma(X, E)$ and $f \in C^\infty(X)$.

Remark 10. If e is a local frame of E on U, we have

$$D^2(e) = D(eA) = De \wedge A + edA = e(A \wedge A + dA) = eF_A,$$

for $F_A = dA + A \wedge A$ a matrix of 2-forms called the ***curvature*** 2***-form*** of D on U.

Exercise 2. Let E_1 and E_2 be two vector bundles, and consider an isomorphism $g : E_1 \to E_2$. Fix two local frames e_1 and e_2 of E_1 and E_2, respectively, on U, and consider the associated matrix $g_U : U \to \mathrm{GL}_r(\mathbb{C})$. Let D_1 be a connection on E_1 and consider the "gauge transformed" connection $D_2 = f \circ D_1 \circ f^{-1}$ on E_2. Let A_1 and A_2 denote the corresponding connection 1-forms on U, of D_1 and D_2 with respect to e_1 and e_2.

Show that

$$A_2 = g_U A_1 g_U^{-1} + g_U d g_U^{-1},$$

and

$$F_{A_2} = g_U F_{A_1} g_U^{-1}.$$

In particular, this implies that the locally defined F_A determine a globally defined $\mathrm{End}\, E$-valued 2-form $F_D \in \Omega^2(X, \mathrm{End}\, E)$.

Exercise 3 (Distributions and connections)**.** A ***distribution*** over a smooth manifold M of dimension n is a subbundle $\Xi \subset TM$ of the tangent bundle of M. Consider now the natural projection $p : E \to X$ of a smooth vector bundle over X. Its differential determines a natural morphism $TE \to p^*TX$ of vector bundles over E. The kernel of this map is the bundle $V_E := p^*E \subset TE$, which we call the ***vertical distribution***.

1. *Prove that a connection D on E determines a* ***horizontal distribution****, namely, that it determines a distribution $H_D \subset TE$ such that*

$$TE = V_E \oplus H_D.$$

A distribution is ***involutive*** if, for any two sections of it (that is, for any two vector field ξ, η which lie on it) their Lie bracket $[\xi, \eta]$ also lies on Ξ.

2. *Prove that H_D is involutive if and only if $D^2 = 0$.*

1.3 Flat bundles and local systems

Definition 11. A connection D on a smooth vector bundle E is ***flat*** if its curvature is 0. A pair (E, D) formed by a smooth vector bundle and a flat connection is called a ***flat bundle***.

If (E_1, D_1) and (E_2, D_2) are two flat bundles, a ***morphism of flat bundles*** $g : (E_1, D_1) \to (E_2, D_2)$ is determined by a morphism of bundles $g : E_1 \to E_2$ such that $D_2 = g \circ D_1 \circ g^{-1}$.

Theorem 12 (Frobenius)**.** *Let (E, D) be a flat bundle. Suppose that E is determined by a cocycle (g_{UV}). Then there exists a 0-coboundary (f_U) such that the functions $g'_{UV} = (f_V)^{-1} g_{UV} f_U$ are locally constant. The corresponding local system E' determined by (g'_{UV}) is called the* ***holonomy local system*** *associated with (E, D).*

Proof. Suppose that we can find, for each $U \in \mathfrak{U}$, a frame ϵ_U of E on U such that $D\epsilon_U = 0$. If we start from the family of frames $\{e_U : U \in \mathfrak{U}\}$ determining E in terms of the cocycle (g_{UV}), each ϵ_U is of the form $\epsilon_U = e_U f_U$, for $f_U : U \to \mathrm{GL}_n(\mathbb{C})$. Now, on a non-empty overlap $U \cap V$, putting $g'_{UV} = f_V^{-1} g_{UV} f_U$, we have

$$0 = D\epsilon_U = D(e_U f_U) = D(e_V g_{UV} f_U) = D(\epsilon_V g'_{UV}) = D\epsilon_V g'_{UV} + \epsilon_V dg'_{UV}.$$

We conclude that $dg'_{UV} = 0$ and thus the g'_{UV} are locally constant.

It remains to see that we can find such a frame ϵ_U. Equivalently, we want to find matrix-valued functions $f_U : U \to \mathrm{GL}_n(\mathbb{C})$ satisfying

$$0 = D(e_U f_U) = D(e_U) f_U + e_U df_U = e_U(Af_U + df_U),$$

where A is the connection 1-form in the frame e_U. Therefore, our problem is reduced to that of finding solutions f to the differential equation

$$df + Af = 0.$$

As we explain in Exercise 4, this is just an application of Frobenius theorem, where the integrability condition corresponds precisely to $F_A = dA + A \wedge A = 0$. □

Exercise 4 (The Frobenius theorem. Analysts version)**.** Consider an open subset $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$, where U is a neighborhood of $0 \in \mathbb{R}^m$. Consider a family of C^∞ functions $F_1, \dots, F_m : U \times V \to \mathbb{R}^n$. The theorem of Frobenius tells us that, for every $x \in V$, there exists one and only one smooth function $\alpha : W \to V$, defined in a neighborhood W of 0 in $\mathbb{R}^n$, with $\alpha(0) = x$ and solving the PDE

$$\frac{\partial \alpha}{\partial t^i}(t) = F_i(t, \alpha(t)), \text{ for all } t \in W,$$

if and only if there is a neighborhood of $(0,x) \in U \times V$ on which

$$\frac{\partial F_j}{\partial t^i} - \frac{\partial F_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial F_j}{\partial x^k} F_i^k - \sum_{k=1}^{n} \frac{\partial F_i}{\partial x^k} F_j^k = 0,$$

for $i, j = 1, \dots, m$.

Prove that the equation $df + Af = 0$ can be written as the PDE above, and that the integrability condition corresponds to the condition $dA + A \wedge A = 0$.

Exercise 5 (The Frobenius theorem. Geometers version)**.** A distribution D on a smooth manifold M is ***integrable*** if there exists some submanifold $N \subset M$ such that, for any point $p \in N$, we have that $T_pN = D_p$. In that case, we say that N is an integral manifold of D. The "geometers version" of the Frobenius theorem says that a distribution D is integrable if and only if it is involutive.

1. *Prove the geometers version of Frobenius theorem from the "analysts version" from the previous exercise.*

We saw in a previous exercise that a connection D on E determines a horizontal distribution $H_D \subset TE$ and that D is flat if and only if H_D is involutive. Consider the corresponding integral manifold $Y \subset E$.

2. *Show that the natural projection $p : E \to X$ restricts to a local homeomorphism $\pi : Y \to X$.*
3. *Show that this $\pi : Y \to X$ is in fact a covering space.*
4. *Show that the monodromy representation associated with $\pi : Y \to X$ coincides with the monodromy representation associated with the holonomy local system determined by (E, D).*

Exercise 6. We have shown that with any flat bundle (E, D) we can associate a local system E'. We want to show that this can be upgraded to an equivalence of categories. In order to do so, we must show the following.

1. *Show that there is a bijection between the set of morphisms $(E_1, D_1) \to (E_2, D_2)$ of flat bundles and between the set of 0-coboundaries (f_U) such that $g'_{1,UV} = f_U g'_{2,UV} f_V^{-1}$.*
2. *Given a local system E' over X, construct a flat bundle (E, D) such that its holonomy local system is isomorphic to E'.*

Exercise 7 (de Rham theorem for degree 1 cohomology)**.** Consider the trivial line bundle $\mathbb{C}_X := X \times \mathbb{C} \to X$. *Show that the set of equivalence classes of flat connections on $\mathbb{C}_X$ is in natural bijection with the de Rham cohomology group $H^1_{\mathrm{dR}}(X, \mathbb{C})$. Use the correspondence of this section to prove that this group is isomorphic to the singular cohomology group $H^1(X, \mathbb{C})$.*

Exercise 8 (Connections of constant central curvature)**.** Let $\omega_X \in \Omega^2(X)$ be a volume form on X with $\int_X \omega_X = 1$. A connection D on a smooth vector bundle has ***constant central curvature*** if

$$F_D = c \, \mathrm{id}_E \, \omega_X,$$

for some constant $c \in \mathbb{C}^*$. Formally, we can assume that ω_X can get more and more concentrated at a single point $x_1 \neq x_0 \in X$ so that in the limit we obtain a Dirac delta $\omega_C = \delta(x_1)$. In this limit, a connection D of constant central curvature is flat away from x_1, so (E, D) restricts to a flat bundle over $X \setminus \{x_1\}$ and thus determines a representation of the fundamental group

$$\rho : \pi_1(X \setminus x_1, x_0) \to \mathrm{GL}_r(\mathbb{C}).$$

Let σ be a contractible loop in X around x_1 and based in x_0. Show that the representation ρ must map the class of σ to $\exp(c/r)$*.*

1.4 Holomorphic structures as Dolbeault operators

We denote by $K_X = \Omega_X^{1,0}$ the holomorphic cotangent bundle of X and more generally, by $\Omega_X^{p,q}$ the vector bundles whose sections are differential forms of type (p,q). Recall that we have the Dolbeault operator $\bar\partial : \Omega^{p,q}(U) \to \Omega^{p,q+1}(U)$. Holomorphic structures arise naturally by generalizing Dolbeault operators to vector bundles.

Definition 13. Let E be a smooth vector bundle on X. A ***holomorphic structure*** $\bar\partial_{\mathcal{E}}$ on $\mathcal{E}$ is a $\mathbb{C}$-linear operator

$$\bar\partial_{\mathcal{E}} : E \to E \otimes \Omega_X^{0,1}$$

such that

$$\bar\partial_{\mathcal{E}}(fs) = s\,\bar\partial\, f + f\,\bar\partial\, s$$

for every smooth function f on U and every section s of E on U, for any open subset $U \subset X$.

Remark 14. In higher dimensions, to obtain a holomorphic structure one should add the condition that $\bar\partial_{\mathcal{E}}^2 = 0$. However on Riemann surfaces this condition is empty, since $\Omega_X^{0,2} = 0$.

There is an analogue of the Frobenius theorem for holomorphic structures, with a sustantially more difficult proof.

Theorem 15. *Consider a pair $(E, \bar\partial_{\mathcal{E}})$ formed by a smooth vector bundle on X with a holomorphic structure $\bar\partial_{\mathcal{E}}$. Suppose that E is determined by a cocycle (g_{UV}). Then there exists a* 0*-coboundary (f_U) such that the functions $g'_{UV} = (f_V)^{-1} g_{UV} f_U$ are holomorphic.*

Remark 16. Following the same arguments as in the proof of 12. It suffices to show that there exist local frames ϵ_U with $\bar\partial_E\, \epsilon_U$. This problem itself reduces to finding solutions to the equation

$$\bar\partial\, f + Af = 0.$$

We refer the reader to [2, Section 5] for details on the integrability of this equation.

Remark 17. The above theorem tells us that, instead of thinking about a holomorphic vector bundle $\mathcal{E}$, we can think about the pair $(E, \bar\partial_{\mathcal{E}})$ formed by the smooth vector bundle E underlying $\mathcal{E}$ and the holomorphic structure $\bar\partial_{\mathcal{E}}$. This is the typical approach in gauge theory.

Remark 18. We also remark the fact that, if D is a connection on a smooth vector bundle E, then we can take its $(0,1)$ part $\bar\partial_D = D^{0,1}$, which determines a holomorphic structure on E.

1.5 Hermitian metrics and the Chern correspondence

Definition 19. Let E be a smooth vector bundle on X. A ***Hermitian metric*** H on E is determined by a Hermitian product $\langle -, - \rangle_{H,x}$ on each fibre E_x, in such a way that for every open subset $U \subset X$ and for every two sections s and t of E on U, the map

$$\begin{aligned} \langle s, t\rangle_H : U &\longrightarrow \mathbb{C} \\ x &\longmapsto \langle s(x), t(x)\rangle_{H,x} \end{aligned}$$

is smooth. A pair (E, H) formed by a smooth vector bundle and a Hermitian metric is called a ***Hermitian vector bundle***.

Definition 20. Let (E, H) be a Hermitian vector bundle. A connection ∇ on E is ***H-unitary*** if for every two local sections s and t of E and for every vector field of ξ on an open $U \subset X$, we have

$$d\langle s, t\rangle_H(\xi) = \langle \nabla s(\xi), t\rangle_H + \langle s, \nabla t(\xi)\rangle_H.$$

Exercise 9. *Show that if (E, H) is a Hermitian vector bundle and ∇ is a flat H-unitary connection on E, then the monodromy representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}_n(\mathbb{C})$ associated with the corresponding local system factors trhough the unitary group $U(n) \subset \mathrm{GL}_n(\mathbb{C})$.*

Theorem 21. *Let $(E, \bar\partial_E)$ be a holomorphic vector bundle. For every Hermitian metric H on E, there exists a canonical connnection ∇_H on E such that $\bar\partial_E = \nabla_H^{0,1}$. This connection is called the **Chern connection**.*

Proof. Consider a frame $\{e_1, \ldots, e_r\}$ of E over some open subset $U \in \mathfrak{U}$, and assume that this frame is holomorphic; that is, that $\bar\partial_E\, e_i = 0$, for $i = 1, \ldots, r$. Let us consider the functions $h_{ij} = \langle e_i, e_j\rangle_H$. If such a ∇_H exists, then its connection 1-form A with respect to this framing must be of type $(0, 1)$, since we must have $\bar\partial_E = \nabla_H^{0,1}$. But then

$$dh_{ij} = d\langle e_i, e_j\rangle = \sum_k A_i^k h_{kj} + h_{ik}\bar A_j^k,$$

so $\partial h_{ij} = \sum_k A_i^k h_{kj}$ and $\bar\partial\, h_{ij} = \sum_k h_{ik}\bar A_j^k$. Therefore, if we consider the matrix $h = (h_{ij})$ we can just set

$$A = h^{-1}\partial h.$$

□

2 Classifying vector bundles

2.1 Topological classification

With any vector bundle E on X we can associate its ***determinant line bundle***, defined as follows. If E is determined by gluing spaces of the form $E_U = U \times \mathbb{C}^r$ via transition functions $g_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$, then $\det E$ is obtained by gluing the spaces $(\det E)_U = U \times \wedge^r\mathbb{C}^r$ through the transition functions $\det g_{UV} : U \cap V \to \mathbb{C}^*$.

Exercise 10. *Show that any smooth vector bundle E of rank $r > 1$ has a nowhere vanishing global section. Is this true for holomorphic vector bundles?* (Hint: Use transversality).

Using the section s from the exercise above, we obtain an injection $s : \mathbb{C}_X \hookrightarrow E$. Now, any smooth vector bundle admits a Hermitian metric, so we orthogonally decompose $E = s(\mathbb{C}_X) \oplus s(\mathbb{C}_X)^\perp$. Iterating this process, we obtain that E can be written as

$$E = \mathbb{C}_X^{r-1} \oplus L,$$

for some line bundle L. Note however that L must be isomorphic to the determinant line bundle $L \cong \det E$. We conclude the following.

Theorem 22. *A smooth vector bundle on X is determined by its rank and its determinant.*

It thus remains the question of classfying smooth line bundles. Now, recall that these line bundles are classified by the cohomology group $H^1(X, \mathbb{C}^*_X)$, where $\mathbb{C}^*_X$ denotes the sheaf of smooth functions $U \to \mathbb{C}^*$. The exponential exact sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathbb{C}_X \longrightarrow \mathbb{C}^*_X \longrightarrow 0$$

induces an exact sequence

$$H^1(X, \mathbb{C}_X) \longrightarrow H^1(X, \mathbb{C}^*_X) \xrightarrow{\delta} H^2(X, 2\pi i\mathbb{Z}) \longrightarrow H^2(X, \mathbb{C}_X).$$

The existence of smooth partitions of unity implies that $H^i(X, \mathbb{C}_X) = 0$ for $i > 0$, so we obtain an isomorphism $\delta : H^1(X, \mathbb{C}^*_X) \cong H^2(X, 2\pi i\mathbb{Z})$. If L is a smooth line bundle on X represented by a cohomology class $[L] \in H^1(X, \mathbb{C}^*_X)$, we define the ***first Chern class*** of L as

$$c_1(L) = \frac{i}{2\pi}\delta([L]) \in H^2(X, \mathbb{Z}).$$

Recall that integration determines an isomorphism

$$\int_X : H^2(X, \mathbb{Z}) \to \mathbb{Z},\ \alpha \mapsto \int_X \alpha.$$

The ***degree*** of L is the number

$$\deg L = \int_X c_1(L) \in \mathbb{Z}.$$

More generally, if E is a vector bundle, then we define its first Chern class as $c_1(E) = c_1(\det E)$, and its degree as $\deg E = \deg(\det E)$.

Exercise 11 (Chern-Weil theory: Computing Chern classes using curvature)**.** Let E be a smooth vector bundle on X. If D is a connection on E and $F_D \in \Omega^2(X, \operatorname{End} E)$ its curvature. Its trace determines a 2-form $\operatorname{tr}(F_D) \in \Omega^2(X)$, and we can consider its cohomology class $[\operatorname{tr}(F_D)]$.

Prove that

$$c_1(E) = \frac{i}{2\pi}[\operatorname{tr}(F_D)].$$

In particular, this implies that, if E admits a flat connection, then $\deg E = 0$.

To sum up, we conclude the following.

Theorem 23. *Smooth vector bundles on a Riemann surface are classified by their rank and their degree.*

2.2 Holomorphic line bundles: the Jacobian

Let $\mathscr{O}_X$ denote the sheaf of holomorphic functions on X and $\mathscr{O}^*_X$ the sheaf of non-vanishing holomorphic functions. Holomorphic line bundles are classified by the ***Picard group*** $\operatorname{Pic}(X) = H^1(X, \mathscr{O}^*_X)$. In the holomorphic case we also have an exponential exact sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}^*_X \longrightarrow 0,$$

which induces an exact sequence

$$H^1(X, 2\pi i\mathbb{Z}) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X, 2\pi i\mathbb{Z}).$$

Consider the subgroup $\mathrm{Pic}^0(X) = \{[L] \in \mathrm{Pic}(X) : \delta([L]) = 0\}$. It follows from the exact sequence above that $\mathrm{Pic}^0(X)$ is isomorphic to the ***Jacobian*** of X, which is defined as the quotient

$$\mathrm{Jac}(X) = H^1(X, \mathscr{O}_X)/H^1(X, 2\pi i\mathbb{Z}).$$

Exercise 12. *Verify that* $\mathrm{Jac}(X)$ *is an abelian variety of dimension* g*, where* g *is the genus of* X. [Hint: First, you need to convince yourself that $H^1(X, \mathscr{O}_X)$ is a complex vector space of dimension g. This follows either directly from Hodge theory or from GAGA and the fact that X is the analytification of a smooth projective curve (which essentially follows from Hodge theory). Second, you need to verify that there is a Riemann form with respect to the lattice $H^1(X, 2\pi i\mathbb{Z})$. You can construct this form using Poincaré duality.]

Remark 24. Note that this already hints on the complexity of classifying holomorphic vector bundles. While smooth line bundles were simply determined by a number, there is a whole complex manifold worth of isomorphism classes of holomorphic line bundles with the same degree. This is the starting point of the study of moduli spaces of bundles.

2.3 The moduli space of vector bundles on a projective curve

When the genus of X is low, the problem of classifying vector bundles can be solved relatively easily. For example, Grothendieck [4] showed that every holomorphic vector bundle over the Riemann sphere $\mathbb{CP}^1$ can be decomposed as a direct sum of line bundles. For genus 1, Atiyah [1] obtained a explicit description of vector bundles in terms of extensions. The problem gets its full complexity for genus ≥ 2. In higher genus, Seshadri [7] constructed the moduli space of polystable vector bundles. This is a condition arising naturally from Mumford's GIT, and that can be formulated as follows.

Definition 25. The ***slope*** of a vector bundle E on X is the number

$$\mu(E) = \deg E/\mathrm{rk}\ E.$$

A holomorphic vector bundle $\mathcal{E}$ on X is ***stable*** if and only if for every holomorphic subbundle $\mathcal{E}' \subset \mathcal{E}$, we have

$$\mu(\mathcal{E}') < \mu(\mathcal{E}).$$

We say that $\mathcal{E}$ is ***polystable*** if it is either stable or a direct sum of stable vector bundles of slope equal to $\mu(\mathcal{E})$.

Let C be a complex projective curve. The same definitions naturally apply for algebraic vector bundles over C.

Definition 26. The ***moduli problem of stable vector bundles of rank*** r ***and degree*** d ***on*** C is the functor

$$\mathbf{N}^s_{r,d} : (\mathbb{C}\text{-schemes})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

which maps a $\mathbb{C}$-scheme S to the set of isomorphism classes of flat families of stable vector bundles of rank r and degree d on C parametrized by S, and a morphism $S \to T$ to the map sending a family to its pull-back.

A ***coarse moduli space*** for $\mathbf{N}^s_{r,d}$ is a $\mathbb{C}$-scheme M with a morphism of functors $\Psi : \mathbf{N}^s_{r,d} \to \mathrm{Hom}(-, M)$ such that

1. $\Psi(\mathbb{C}) : \mathbf{N}^s_{rd}(\mathbb{C}) \to M(\mathbb{C})$ is a bijection,
2. for every $\mathbb{C}$-scheme M' and any morphism of functors $\Psi' : \mathbf{N}^s_{r,d} \to \mathrm{Hom}(-, M')$, there exists a unique morphism $f : M \to M'$ such that the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Hom}(-, M) \\ & \overset{\Psi}{\nearrow} & \\ \mathbf{N}^s_{r,d} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(-,f)} \\ & \underset{\Psi'}{\searrow} & \\ & & \mathrm{Hom}(-, M'). \end{array}$$

Theorem 27 (Seshadri)**.** *Let C be a complex projective curve of genus $g \geq 2$. There exists a projective variety $\mathcal{N}_{r,d}$, the* ***moduli space of polystable vector bundles of rank*** *r* ***and degree*** *d on C, such that:*

1. *The set closed points $\mathcal{N}_{r,d}(\mathbb{C})$ is in natural bijection with the set of isomorphism classes of polystable algebraic vector bundles of rank r and degree d on C.*
2. *There is a Zariski open subvariety $\mathcal{N}^s_{r,d} \subset \mathcal{N}_{r,d}$ which is a coarse moduli space for the moduli problem $\mathbf{N}^s_{r,d}$.*

2.4 Analytic construction of the moduli space

Let E be a smooth vector bundle. We denote by $\mathcal{C}_E$ the space of holomorphic structures $\bar{\partial}_{\mathcal{E}}$ on E. The difference of any two holomorphic structure is a $(1,0)$-form valued in in $\mathrm{End}\, E$. Therefore, the space $\mathcal{C}_E$ is an affine space modelled by the infinite dimensional vector space $\Omega^{0,1}(X, \mathrm{End}\, E)$. The ***complex gauge group*** $\mathcal{G}^{\mathbb{C}}_E = \Omega^0(X, \mathrm{Aut}\, E)$ acts on $\mathcal{C}_E$ by conjugation

$$g \cdot \bar{\partial}_{\mathcal{E}} = g\, \bar{\partial}_{\mathcal{E}}\, g^{-1}.$$

Note that for such a $g \in \mathcal{G}^{\mathbb{C}}_E$, the holomorphic vector bundles $(E, \bar{\partial}_{\mathcal{E}})$ and $(E, g \cdot \bar{\partial}_{\mathcal{E}})$ are isomorphic. Conversely, two holomorphic vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if and only if their associated operators $\bar{\partial}_{\mathcal{E}}$ and $\bar{\partial}_{\mathcal{E}'}$ are related by some $g \in \mathcal{G}^{\mathbb{C}}_E$.

The quotient set $\mathcal{C}_E / \mathcal{G}^{\mathbb{C}}_E$ is therefore the set of isomorphism classes of holomorphic vector bundles with underlying smooth bundle E. If we restrict to the subset $\mathcal{C}^{ps}_E \subset \mathcal{C}_E$ of holomorphic structures $\bar{\partial}_{\mathcal{E}}$ such that $\mathcal{E}$ is polystable, then $\mathcal{C}^{ps}_E / \mathcal{G}^{\mathbb{C}}_E$ is the set of isomorphism classes of polystable holomorphic vector bundles. Taking care of the common functional-analysis technicalities, one can endow the quotient $\mathcal{C}^{ps}_E / \mathcal{G}^{\mathbb{C}}_E$ with the structure of an analytic space, naturally coming from the fact that it is a quotient of an open subset in an (infinite-dimensional) vector space by an (infinite-dimensional) complex Lie group. This quotient is in fact the analytification of the "algebraic" moduli space $\mathcal{N}_{r,d}$.

2.5 Hermitian-Einstein metrics

We consider once again our Riemann surface X. Let us fix a volume form $\omega_X \in \Omega^2(X)$ on X with $\int_X \omega_X = 1$. Polystability is related to the existence of a special type of metrics.

Definition 28. A ***Hermitian-Einstein*** metric (HE metric) on a holomorphic vector bundle $\mathcal{E}$ is a Hermitian metric H on $\mathcal{E}$ such that its Chern connection ∇_H has constant central curvature; that is, such that

$$F_H = c\,\mathrm{id}_{\mathcal{E}}\,\omega_X,$$

for $F_H = F_{\nabla_H}$ and some constant $c \in \mathbb{C}^*$.

Exercise 13. *Show that if H is a HE metric on $\mathcal{E}$, then c must be equal to $-2\pi i\mu(\mathcal{E})$.*

Proposition 29. *If $\mathcal{E}$ admits a HE metric then it is polystable.*

Proof. Suppose that $\mathcal{E}' \subset \mathcal{E}$ is a holomorphic subbundle of $\mathcal{E}$ and consider the quotient $\mathcal{E}'' = \mathcal{E}/\mathcal{E}'$. We can write

$$\nabla_H = \begin{pmatrix} \nabla' & \beta \\ -\beta^\dagger & \nabla'' \end{pmatrix}.$$

Here, ∇' and ∇'' are the restriction and the projection of ∇_H to $\mathcal{E}'$ and $\mathcal{E}''$, respectively, while $\beta \in \Omega^{0,1}(X, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'))$ is a representative of the class of $\mathcal{E}$ as extension of $\mathcal{E}''$ by $\mathcal{E}'$. In particular, if $\beta = 0$, then $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$. The form $\beta^\dagger \in \Omega^{1,0}(X, \mathrm{Hom}(\mathcal{E}', \mathcal{E}''))$ is just the conjugate transpose of β.

Now, the top left element of F_H is $F_{\nabla'} - \beta \wedge \beta^\dagger$. Taking traces, integrating and multiplying by $\frac{i}{2\pi}$, we obtain

$$c\frac{i}{2\pi}\mathrm{rk}\ \mathcal{E}' = \frac{i}{2\pi}\int_X \mathrm{tr}\ F_{\nabla'} + \|\beta\|^2.$$

From here, we get

$$\mu(\mathcal{E}) = \mu(\mathcal{E}') + C\|\beta\|^2,$$

for some constant $C > 0$. Therefore, $\mu(\mathcal{E}) \geq \mu(\mathcal{E}')$, with equality if and only if $\beta = 0$. □

2.6 The theorem of Narasimhan–Seshadri

The converse of the above proposition is the celebrated theorem of Narasimhan–Seshadri [5] (see also [2, 3]).

Theorem 30 (Narasimhan–Seshadri)**.** *Every polystable holomorphic vector bundle admits a HE metric.*

This result can be interpreted in a more "structural" way as follows. Let us start by consdering a smooth vector bundle E. We denote by $\mathcal{A}_E$ the space of connections on E. This is an affine space modelled over the (infinite-dimensional) vector space $\Omega^1(X, \mathrm{End}\, E)$. If H is a Hermitian metric on E, we can consider the subspace $\mathcal{A}_{E,H} \subset \mathcal{A}_E$ of connections which are H-unitary. This is an affine subspace, modelled by the vector space $\Omega^1(X, \mathfrak{u}_H E)$, where $\mathfrak{u}_H E$ is the subspace of endomorphisms which are skew-Hermitian (that is $f^\dagger = -f$) with respect to the metric H. The space $\mathcal{A}_{E,H}$ is acted on by the ***gauge group***

$\mathcal{G}_{E,H} = \Omega^0(X, U_H E)$, for $U_H E$ the subgroup of H-unitary automorphisms. Consider the map

$$\mu : \mathcal{A}_{E,H} \to \Omega^2(X, \mathfrak{u}_H E),\ \nabla \mapsto F_\nabla$$

sending each unitary connection to its curvature. The subspace $\mathcal{A}^0_{E,H} \subset \mathcal{A}_{E,H}$ of connections of central constant curvature is the preimage $\mu^{-1}(-2\pi i \mu(E)\, \mathrm{id}_E\, \omega_X)$. The quotient $\mathcal{A}^0_{E,H}/\mathcal{G}_E$ is the ***d*-twisted $U(r)$-character variety**

$$\mathcal{X}^d_{U(r)} = \left\{ (A_1, \ldots, A_g, B_1, \ldots, B_g, Z) \in U(r)^{2g+1} : \prod_{i=1}^{g} [A_i, B_i] = Z, Z = e^{-\frac{2\pi i d}{r}} I_r \right\} /U(r),$$

where $U(r)$ acts by conjugation.

The theorem of Narasimhan–Seshadri is then telling us that the map

$$\mathcal{C}_E \to \mathcal{A}_{E,H}, \bar{\partial}_{\mathcal{E}} \mapsto \nabla_{(\mathcal{E},H)},$$

sending a holomorphic structure $\bar{\partial}_{\mathcal{E}}$ to the Chern connection of the Hermitian holomorphic bundle $(\mathcal{E}, H)$, descends to a map $\mathcal{C}^s_E \to \mathcal{A}^0_{E,H}$ and induces a natural bijection

$$\mathcal{N}_{r,d}(\mathbb{C}) \cong \mathcal{X}^d_{U(r)}.$$

Remark 31. In fact, the theorem of Narasimhan-Seshadri not only gives a bijection, but actually a homeomorphism, which restricts to a diffeomorphism on the smooth parts. The way to convince ourselves that this is true is to study the associated ***deformation complexes***. This is beyond the scope of these notes, but we refer the interested reader to [5, 8, 9].

3 Nonabelian Hodge theory

3.1 Character varieties

Let $\Gamma = \langle s_1, \ldots, s_p : r_1(s_1, \ldots, s_p) = 1, \ldots, r_q(s_1, \ldots, s_q) = 1 \rangle$ be a finitely presented group. The **GL_r-*representation variety*** $\mathcal{R}_{\Gamma,\mathrm{GL}_r}$ (over $\mathbb{C}$) associated with Γ is the affine variety representing by the functor sending any $\mathbb{C}$-algebra A to the set

$$\begin{aligned}\mathcal{R}_{\Gamma,\mathrm{GL}_r}(A) &= \mathrm{Hom}(\Gamma, \mathrm{GL}_r(A)) \\ &= \{S_1, \ldots, S_p \in \mathrm{GL}_r(A) : r_1(S_1, \ldots, S_p) = I_r, \ldots, r_q(S_1, \ldots, S_p) = I_r\}.\end{aligned}$$

The group GL_r acts on $\mathcal{R}_{\Gamma,\mathrm{GL}_r}$ by conjugation and the GIT quotient

$$\mathcal{X}_{\Gamma,\mathrm{GL}_r} = \mathcal{R}_{\Gamma,\mathrm{GL}_r} /\!\!/ \mathrm{GL}_r = \mathrm{Spec}(\mathbb{C}[\mathcal{R}_{\Gamma,\mathrm{GL}_r}]^{\mathrm{GL}_r})$$

is called the **GL_r-*character variety*** (over $\mathbb{C}$) associated with Γ. More generally, if we fix a generator $s_i \in \Gamma$ conjugacy class $c \subset \mathrm{GL}_r$, we can also consider the subvariety $\mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r} \subset \mathcal{R}_{\Gamma,\mathrm{GL}_r}$ representing the functor

$$A \mapsto \mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r}(A) = \{\rho : \Gamma \to \mathrm{GL}_r(A) : \rho(s_i) \in c\},$$

and the corresponding GIT quotient

$$\mathcal{X}^{c,s_i}_{\Gamma,\mathrm{GL}_r} = \mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r} /\!\!/ \mathrm{GL}_r.$$

One of the important properties of the GIT quotient is that the closed points of $\mathcal{X}^{c,s_i}_{\Gamma,\mathrm{GL}_r}$ correspond to the closed $\mathrm{GL}_r(\mathbb{C})$ orbits in $\mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r}$. Now, these orbits are precisely the orbits of the semisimple representations. Recall that a representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ is ***semisimple*** if and only if it decomposes as a direct sum of simple representations. Therefore, if we consider the subset $\mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r}(\mathbb{C})^+ \subset \mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r}(\mathbb{C})$ consisting of semisimple representations, we have

$$\mathcal{X}^{c,s_i}_{\Gamma,\mathrm{GL}_r}(\mathbb{C}) = \mathcal{R}^{c,s_i}_{\Gamma,\mathrm{GL}_r}(\mathbb{C})^+ / \mathrm{GL}_r(\mathbb{C}).$$

3.2 The Betti moduli space

Let us consider now our compact Riemann surface X, with two marked points x_0 and x_1, and let us take

$$\Gamma = \pi_1(X \setminus \{x_1\}, x_0) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, z : \prod_{i=1}^{g} [a_i, b_i] = z \right\rangle.$$

For any integer d, we let $c_d \subset \mathrm{GL}_r$ denote the conjugacy class of the matrix $e^{-\frac{2\pi i d}{r}} I_r$. We define the ***Betti moduli space*** $\mathcal{M}^B_{r,d}$ of X as

$$\mathcal{M}^B_{r,d} = \mathcal{X}^{c_d,z}_{\Gamma,\mathrm{GL}_r}.$$

In particular, for $d = 0$, we obtain the character variety

$$\mathcal{M}^B_{r,d} = \mathcal{X}_{\pi_1(X,x_0),\mathrm{GL}_r}.$$

We can construct this moduli space analytically as follows. Let E be a smooth vector bundle over X and consider the space $\mathcal{A}_E$ of connections in E. Recall that we have a map

$$\mu : \mathcal{A}_E \to \Omega^2(X, \mathrm{End}\, E), D \mapsto F_D,$$

and we can consider the subspace $\mathcal{A}^0_E = \mu^{-1}(-2\pi i \mu(E) \, \mathrm{id}_E \, \omega_X)$ of connections of constant central curvature. The holonomy representation determines a map

$$\mathcal{A}^0_E \to \mathcal{R}^{c_d,z}_{\Gamma,\mathrm{GL}_r}(\mathbb{C}).$$

Exercise 14. A connection $D \in \mathcal{A}^0_E$ is called ***reductive*** if every D-invariant subbundle of E admits a D-invariant complement. *Prove that D is reductive if and only if the corresponding holonomy representation is semisimple.*

We consider then the subspace $\mathcal{A}^{0,+}_E \subset \mathcal{A}^0_E$ of reductive connections. The holonomy representation then induces a homeomorphism

$$\mathcal{A}^{0,+}_E / \mathcal{G}^{\mathbb{C}}_E \cong \mathcal{R}^{c_d,z}_{\Gamma,\mathrm{GL}_r}(\mathbb{C})^+ / \mathrm{GL}_r(\mathbb{C}) = \mathcal{M}^B_{r,d}(\mathbb{C}).$$

3.3 Harmonic metrics and the Hitchin equations

A Hermitian metric H on a smooth vector bundle E induces a natural "Cartan decomposition"

$$\mathrm{End}\, E = \mathfrak{u}_H E \oplus i\mathfrak{u}_H E,$$

since every endomorphism can be decomposed in its Hermitian and skew-Hermitian parts. Therefore, if D is a connection on E, then we can write

$$D = \nabla + i\Phi,$$

where ∇ is a H-unitary connection on E and $\Phi \in \Omega^1(X, \mathfrak{u}_H E)$.

If the connection D has constant central curvature, then we have

$$-2\pi i\mu(E)\,\mathrm{id}_E\,\omega_X = F_D = F_\nabla - \tfrac{1}{2}[\Phi,\Phi] + \nabla\Phi + \nabla^\dagger\Phi.$$

Here $\nabla^\dagger$ is defined using both the metric on X and the Hermitian metric H. Separating in Hermitian and skew-Hermitian parts, we obtain the equations

$$\begin{cases} F_\nabla - \frac{1}{2}[\Phi,\Phi] = -2\pi i\mu(E)\,\mathrm{id}_E\,\omega_X, \\ \nabla\Phi + \nabla^\dagger\Phi = 0. \end{cases}$$

Definition 32. A Hermitian metric on a pair (E, D) formed by a bundle and a metric with constant central curvature is called ***harmonic*** if

$$\nabla^\dagger\Phi = 0.$$

Definition 33. A ***solution to the Hitchin equations*** is a tuple (E, H, ∇, Φ) formed by a smooth Hermitian vector bundle (E, H), a H-unitary connection ∇ on E and a 1-form $\Phi \in \Omega^1(X, \mathfrak{u}_H E)$ such that it satisfies the ***Hitchin equations***

$$\begin{cases} F_\nabla - \frac{1}{2}[\Phi,\Phi] = -2\pi i\mu(E)\,\mathrm{id}_E\,\omega_X, \\ \nabla\Phi = \nabla^\dagger\Phi = 0. \end{cases}$$

Clearly, H is a harmonic metric on a pair (E, D) if and only if the corresponding (E, H, ∇, Φ) is a solution to the Hitchin equations. The crucial result is then the following.

Theorem 34 (Donaldson–Corlette)**.** *A pair (E, D) admits a harmonic metric if and only if the connection D is reductive.*

Let us fix now any Hermitian metric H on E, and consider the space

$$\mathcal{H}_{E,H} = \mathcal{A}_{E,H} \oplus \Omega^1(X, \mathfrak{u}_H E)$$

and the subspace $\mathcal{H}^0_{E,H} \subset \mathcal{H}_{E,H}$ consisting of pairs (∇, Φ) such that (E, H, ∇, Φ) is a solution to the Hitchin equations. The gauge group $\mathcal{G}_{E,H}$ acts naturally on $\mathcal{H}_{E,H}$ preserving the subspace $\mathcal{H}^0_{E,H}$. The quotient

$$\mathcal{M}^H_{r,d} = \mathcal{H}^0_{E,H}/\mathcal{G}_{E,H}$$

is called the ***moduli space of solutions to the Hitchin equations*** or just the ***Hitchin moduli space***.

The theorem of Donaldson–Corlette is then telling us that the map

$$\mathcal{A}^0_E \to \mathcal{H}_{E,H}, D \mapsto (\nabla, \Phi)$$

descends to a map $\mathcal{A}^{0,+}_E \to \mathcal{H}^0_{E,H}$ and induces a homeomorphism

$$(\mathcal{M}^B_{r,d})^{\mathrm{an}} \cong \mathcal{M}^H_{r,d}.$$

3.4 Higgs bundles over a complex projective curve

Definition 35. A ***Higgs bundle*** is a pair $(\mathcal{E}, \varphi)$ consisting of a holomorphic vector bundle $\mathcal{E}$ over X a and holomorphic section $\varphi \in H^0(X, \operatorname{End} \mathcal{E} \otimes K_X)$.

Remark 36. One should think (locally) about φ as a matrix of $(1,0)$-forms.

The same definition clearly applies over a complex projective curve C, replacing $\mathcal{E}$ by an algebraic vector bundle, K_X by the algebraic cotangent bundle of C, and φ by an algebraic section.

Definition 37. A Higgs bundles $(\mathcal{E}, \varphi)$ is ***stable*** if and only if for every φ-invariant holomorphic subbundle $\mathcal{E}' \subset \mathcal{E}$, we have

$$\mu(\mathcal{E}') < \mu(\mathcal{E}).$$

We say that $\mathcal{E}$ is ***polystable*** if it is either stable or a direct sum of stable Higgs bundles with slope equal to $\mu(\mathcal{E})$.

As we did for vector bundles, we can similarly define the moduli problem and the notion of coarse moduli space of stable Higgs bundles such that the underlying bundle has fixed rank r and degree d. We denote this moduli problem by $\mathbf{M}^s_{r,d}$.

Theorem 38 (Nitsure)**.** *Let C be a complex projective curve of genus $g \geq 2$. There exists a quasi-projective variety $\mathcal{M}_{r,d}$, called the* ***moduli space of polystable Higgs bundles of rank r and degree d on C****, or just* ***Dolbeault moduli space****, such that:*

1. *The set of closed points $\mathcal{M}_{r,d}(\mathbb{C})$ is in natural bijection with the set of isomorphism classes of polystable algebraic Higgs bundles of rank r and degree d on C.*

2. *There is a Zariski open subvariety $\mathcal{M}^s_{r,d} \subset \mathcal{M}_{r,d}$ which is a coarse moduli space for $\mathbf{M}^s_{r,d}$.*

The analytification of the Dolbeault moduli space is a complex manifold parametrizing (holomorphic) Higgs bundles on $X = C^{\mathrm{an}}$. It admits an analytic construction similar to that of the moduli space of vector bundles. Let E be a smooth vector bundle. We denote by $\mathcal{S}_E \subset \mathcal{C}_E \times \Omega^0(X, \operatorname{End} E \otimes K_X)$ the subspace of pairs $(\bar\partial_{\mathcal{E}}, \varphi)$ such that $\bar\partial_{\mathcal{E}} \varphi = 0$. In other words, these are pairs formed by a holomorphic structure on E and a section of $\operatorname{End} E \otimes K_X$ which is holomorphic with respect to that holomorphic structure. Thus, a Higgs bundle $(\mathcal{E}, \varphi)$ with underlying smooth vector bundle E can be regarded simply as a pair $(\bar\partial_{\mathcal{E}}, \varphi) \in \mathcal{S}_E$. The complex gauge group $\mathcal{G}^{\mathbb{C}}_E = \Omega^0(X, \operatorname{Aut} E)$ acts on $\mathcal{S}_E$ by conjugation

$$g \cdot (\bar\partial_{\mathcal{E}}, \varphi) = (g\, \bar\partial_{\mathcal{E}}\, g^{-1}, g\varphi g^{-1}),$$

and it is clear that two pairs $(\bar\partial_{\mathcal{E}}, \varphi)$ and $(\bar\partial_{\mathcal{E}'}, \varphi')$ are related by some $g \in \mathcal{G}^{\mathbb{C}}_E$ if and only if the corresponding Higgs bundles $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ are isomorphic. Therefore the quotient $\mathcal{S}_E/\mathcal{G}^{\mathbb{C}}_E$ is the set of isomorphism classes of Higgs bundles with underlying smooth bundle E. If we restrict to the subset $\mathcal{S}^{ps}_E \subset \mathcal{S}$ of pairs $(\bar\partial_{\mathcal{E}}, \varphi)$ such that $(\mathcal{E}, \varphi)$ is polystable, then $\mathcal{S}^{ps}_E/\mathcal{G}^{\mathbb{C}}_E$ is the set of isomorphism classes of stable Higgs bundles. This quotient can be endowed with the structure of an analytic space, and it coincides with the analytification of the "algebraic" moduli space $\mathcal{M}_{r,d}$.

Consider now a Higgs bundle $(\mathcal{E}, \varphi)$, and let H be a Hermitian metric on $\mathcal{E}$. There is then an associated Chern connection ∇_H. We can also construct a 1-form $\Phi_H \in \Omega^1(X, \mathfrak{u}_H E)$ by putting

$$\Phi_H = \varphi - \varphi^\dagger.$$

Exercise 15. *Show that the equation $\bar{\partial}_{\mathcal{E}}\, \varphi = 0$ implies that*

$$\nabla_H \Phi_H = \nabla_H^{\dagger} \Phi_H = 0.$$

Definition 39. A ***Hermitian-Einstein-Higgs*** metric (HEH metric) on a Higgs bundle $(\mathcal{E}, \varphi)$ is a Hermitian metric H on $\mathcal{E}$ such that the associated pair (∇_H, Φ_H) defined above satisfies the equation

$$F_H - \frac{1}{2}[\Phi, \Phi] = -2\pi i \mu(E)\, \mathrm{id}_E\, \omega_X.$$

In other words, H is a HEH metric on $(\mathcal{E}, \varphi)$ if and only if (E, H, ∇_H, Φ_H) is a solution to the Hitchin equations.

Theorem 40 (Hitchin–Simpson)**.** *A Higgs bundle $(\mathcal{E}, \varphi)$ admits a HEH metric if and only if it is polystable.*

This theorem tells us that, if we fix a smooth vector bundle E and a Hermitian metric H on it, the map

$$\mathcal{S}_E \to \mathcal{H}_{E,H}, (\bar{\partial}_{\mathcal{E}}, \varphi) \mapsto (\nabla_H, \Phi_H)$$

desdends to a map $\mathcal{S}_E^{ps} \to \mathcal{H}_{E,H}^0$ and induces a homeomorphism

$$(\mathcal{M}_{r,d})^{\mathrm{an}} \cong \mathcal{M}_{r,d}^H.$$

Exercise 16 (Some examples of Higgs bundles)**.** *Can you think of any "trivial" or easy examples of Higgs bundles. Are they stable?*

A less trivial example is obtained if we consider any holomorphic line bundle $\mathcal{L}$ over X and take $\mathcal{E} = \mathcal{L} \otimes K_X \oplus \mathcal{L}$. For any pair of sections $(a_1, a_2) \in H^0(X, K_X) \oplus H^0(X, K_X^2)$, we can equip $\mathcal{E}$ with the Higgs field

$$\varphi = \begin{pmatrix} a_1 & a_2 \\ 1 & a_1 \end{pmatrix}.$$

Show that, despite the fact that $\mathcal{E}$ is not stable nor polystable, the Higgs bundle $(\mathcal{E}, \varphi)$ is indeed stable.

Exercise 17 (Spin structures, and some more examples)**.** A ***spin structure*** or ***theta-characteristic*** on X is a holomorphic line bundle $\mathcal{L}$ on X such that $\mathcal{L}^2 \cong K_X$.

Show that the set of spin structures on X up to equivalence is a torsor under the cohomology group $H^1(X, \mathbb{Z}/2\mathbb{Z})$. Therefore, there are exactly 2^{2g} equivalent spin structures on a genus g surface. Why do you think these are called spin structures?

Fix a spin structure $\mathcal{L}$ on X and consider the holomorphic vector bundle

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}.$$

For any $a_2 \in H^0(X, K_X^2)$, we can equip $\mathcal{E}$ with the Higgs field

$$\varphi = \begin{pmatrix} 0 & a_2 \\ 1 & 0 \end{pmatrix}.$$

In particular, note that $\det \mathcal{E} = \mathcal{O}_X$ and that $\mathrm{tr}(\varphi) = 0$. This is what is called an SL_2-Higgs bundle.

3.5 Summary of nonabelian Hodge theory

Let us pause for a second to summarize the main statements of nonabelian Hodge theory. Let E be a smooth vector bundle on X of rank r and degree d.

1. If $(\mathcal{E}, \varphi)$ is a polystable Higgs bundle on X, we can find a HEH metric on it, and obtain a flat connection
$$D = \partial_{\mathcal{E}}^{H} + \bar{\partial}_{\mathcal{E}} + \varphi - \varphi^{\dagger}.$$
2. If (E, D) is a reductive flat bundle on X and H is a harmonic metric on it, then we can obtain a Higgs bundle
$$(\mathcal{E}, \varphi) = ((E, \nabla^{0,1}), \tfrac{1}{2}\Phi),$$
where $D = \nabla + i\Phi$ is the decomposition into Hermitian and skew-Hermitian parts induced by H.
3. The above determines a bijection, and in fact a homeomorphism between (the analytifications of) the Betti moduli space $\mathcal{M}_{r,d}^{B}$ and the Dolbeault moduli space $\mathcal{M}_{r,d}$.

3.6 Profitting from nonabelian Hodge theory

Exercise 18 (Uniformization à la Hitchin)**.** Nonabelian Hodge theory is so strong that it implies the uniformization theorem. Let us explore this in detail. We start by fixing a Riemannian metric $g = u(z, \bar{z})dzd\bar{z}$ compatible with the complex structure of X (that is, compatible with the conformal structure). The Levi-Civita connection associated to this metric can be regarded as a $U(1)$-connection on the canonical line bundle K_X. The curvature F_0 of the metric g is the curvature of the induced $U(1)$-connection on the tangent bundle K_X^{-1}.

Let us now fix a spin structure $\mathcal{L}$ on X with the induced $U(1)$-connection. In turn we obtain a connection (reducible to $U(1)$) on the vector bundle $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}$, with curvature
$$F = \begin{pmatrix} -\frac{1}{2}F_0 & 0 \\ 0 & \frac{1}{2}F_0 \end{pmatrix}.$$

Consider the Higgs field
$$\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We already now that this Higgs bundle $(\mathcal{E}, \varphi)$ is stable (it is a particular case of Exercise 16). The Hitchin equation then becomes
$$\begin{pmatrix} -\frac{1}{2}F_0 & 0 \\ 0 & \frac{1}{2}F_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} udzd\bar{z}.$$

Therefore, we obtain the equation
$$F_0 = -2g.$$

Verify that this means precisely that g has constant curvature equal to -4. Conclude from here the uniformization theorem.

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