Costa Rica - Workshop on Character Varieties and Higgs Bundles

Character Varieties

Ana Casimiro

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Abstract

These notes are based on the material presented in one of the mini-courses of the Workshop on Character Varieties and Higgs Bundles, held in Liberia, Guanacaste, Costa Rica, in August 2025.

We begin by recalling the notions of real or complex Lie and algebraic groups, as well as finitely generated groups, essential ingredients in the definition of a character variety. We then examine the space of representations of a finitely generated group into a Lie or algebraic group, discussing its topology, variety structure, tangent space, and the conjugation action of the target group. Next, we construct the quotient by this action, the character variety, and present different approaches to this construction. We also explore the existence of deformation retractions between character varieties when considering maximal compact subgroups of the Lie group. Finally, we discuss the correspondence between representations of surface groups and principal bundles, with a particular focus on Schottky representations.

Contents

1 Preliminaries

In this section, we briefly review Lie groups and algebraic groups, as well as some basic notions about finitely generated groups. These are the main ingredients in the construction of character varieties. Our exposition follows the insightful survey by Arnaut Maret [Mar25].

1.1 Lie groups

Definition 1.1. A Lie group G is a real/complex smooth manifold with a group structure, where the group operations are smooth/holomorphic maps.

Every Lie group admits an analytic atlas (unique up to diffeomorphism), such that the group operations are analytic maps.

Let G° denote the identity component of a Lie group G, the centralizer of a subset S of G, $Z(S)$, is the following Lie subgroup of G

$$Z(S) := \{g \in G : gsg^{-1} = s, \forall s \in S\}$$

When $S = G$, $Z(G)$ is called the center of G.

Next, we give some classical examples of matrix or linear Lie groups, which will be the cases we will consider.

Example 1.2.

1. The group of invertible $n \times n$ matrices denoted by $\mathrm{Gl}(n, \mathbb{R})$ or $\mathrm{Gl}(n, \mathbb{C})$, for the real or complex case, respectively, called general linear groups.

2. The subgroups of the previous examples, $\mathrm{Sl}(n, \mathbb{R}) = \{M \in \mathrm{Gl}(n, \mathbb{R}) : \det(M) = 1\}$ (special linear group), $\mathrm{O}(n) = \{M \in \mathrm{Gl}(n, \mathbb{R}) : M^\top M = I_n\}$ (orthogonal group), $\mathrm{SO}(n, \mathbb{R}) = \mathrm{O}(n) \cap \mathrm{Sl}(n, \mathbb{R})$ (special orthogonal group), $\mathrm{SU}(p.q)$, $\mathrm{Sp}(2n, \mathbb{R})$ for the real case and $\mathrm{Sl}(n, \mathbb{C})$ (special linear group), $\mathrm{SO}(n, \mathbb{C})$ for the complex case (of course, these also have a real structure).

There are Lie groups that are not linear, an example is the universal cover of $\mathrm{Sl}(2,\mathbb{R})$.

Performing the quotient of a Lie group G by its center, we get a Lie group, called the *adjoint Lie group* of G. For the linear group case, it is usually added a P before the group name.

The Lie algebra of a Lie group G will be denoted by $\mathfrak{g}$, this can be characterized as the tangent space of G at the identity element. There exists the so-called *exponential map* $\exp : \mathfrak{g} \to G$. An important map is the *adjoint representation* of G on $\mathfrak{g}$, defined by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ such that

$$\mathrm{Ad}(g)(v) = \frac{d}{dt}\bigg|_{t=0} g\exp(tv)g^{-1}, \qquad g \in G,\ v \in \mathfrak{g}$$

where $\mathrm{Aut}(\mathfrak{g})$ is the group of automorphims of $\mathfrak{g}$.

Considering its derivative at the identity element of G, which is called the *adjoint representation* of $\mathfrak{g}$ and is denoted by ad, it is a linear map between $\mathfrak{g}$ and its space of endomorphisms, $\mathrm{End}(\mathfrak{g})$. If the Lie algebra $\mathfrak{g}$ has the Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, then

$$\mathrm{ad}(v_1)(v_2) = [v_1, v_2], \quad v_1, v_2 \in \mathfrak{g}$$

The kernel of ad is called the center of $\mathfrak{g}$ and is denoted by $\mathfrak{z}(\mathfrak{g})$. This can be characterized as the Lie algebra of $Z(G)$.

A Lie algebra $\mathfrak{g}$ can be classified as follows:

1. A *simple Lie algebra* is a non-abelian Lie algebra with the zero ideal as its only proper ideal. Using the one-to-one correspondence between the ideal of a Lie algebra and subrepresentations of its adjoint representation, a Lie algebra is simple if and only if its adjoint representation is irreducible and it is not a 1-dimensional abelian Lie algebra.

2. A *semisimple Lie algebra* is a Lie algebra with only zero abelian ideals. This is equivalent to be a direct sum of simple Lie algebras. Using the so called Killing form for a Lie algebra $\mathfrak{g}$,

$$\begin{aligned} K : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{R} \\ (v_1, v_2) &\longmapsto \mathrm{Tr}(\mathrm{ad}(v_1)\mathrm{ad}(v_2)) \end{aligned}$$

   By the Cartan criterion, a Lie algebra is semisimple if and only if its Killing form is non-degenerate.

3. A *reductive Lie algebra* is a direct sum of an abelian and a semisimple Lie algebra. This is equivalent to its adjoint representation being completely reducible, that is, it decomposes as a direct sum of irreducible representations. Another equivalent characterization is that a reductive Lie algebra admits a faithful, completely reducible, finite-dimensional representation.

A simple Lie algebra is semisimple, and the semisimple one is a reductive Lie algebra.

A Lie group is called *simple*, *semisimple* or *reductive* if its Lie algebra is simple, semisimple or reductive, respectively.

Example 1.3.

1. The groups $\mathrm{Sl}(n,\mathbb{R})$, for $n \geq 2$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SU}(p,q)$, for $p+q \geq 2$ are simple. The identity component of the group $\mathrm{SO}(n)$ is simple for $n \geq 3$ and $n \neq 4$ and semisimple for $n = 4$.

2. The identity component of the group $\mathrm{Gl}(n,\mathbb{R})$ is not semisimple for $n \geq 1$, but it is reductive. Its Lie algebra is the direct sum of the simple Lie algebras of traceless matrices and the abelian Lie algebra of diagonal matrices.

   Another characterization to be reductive is that a connected linear Lie subgroup G of $\mathrm{Gl}(n,\mathbb{R})$ is reductive if and only if the trace form

$$\begin{aligned} \mathrm{Tr} : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{R} \\ (v_1, v_2) &\longmapsto \mathrm{Tr}(v_1 v_2) \end{aligned}$$

   is non-degenerate. This result also holds for the complex case and it will induce a non-degenerate, symmetric, Ad-invariant, real-valued bilinear form given by the real part of the trace form, denoted by $\mathfrak{R}(\mathrm{Tr})$.

3. Consider $\mathrm{Sl}(2,\mathbb{R})$ and the trace form

$$\begin{aligned} \mathrm{Tr} : \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R} &\longrightarrow \mathbb{R} \\ (v_1, v_2) &\longmapsto \mathrm{Tr}(v_1 v_2) \end{aligned}$$

   The trace of a matrix is invariant under conjugation, then the trace will be Ad-invariant. Choosing the following basis for $\mathfrak{sl}_2\mathbb{R}$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

the trace form is given by $2x_1x_2 + y_1z_2 + z_1y_2$ which is a symmetric and non-degenerate with signature $(2, 1)$.

1.2 Algebraic groups

Some good references on Algebraic groups are [**Borel**; Hum75; Mil13; Mil17].

Definition 1.4. An *algebraic group* G is an (affine) algebraic variety, that is, the zero locus of a set of polynomials over $\mathbb{R}$ or $\mathbb{C}$, with a group structure where operations maps are regular algebraic maps (restrictions of polynomials maps).

The Zariski closure of a subgroup of an algebraic group is also an algebraic group and an algebraic subgroup of an algebraic group is Zariski closed. An application of this is that the centralizer of a subset of an algebraic group is Zariski colsed, then it is an algebraic group. All the real or complex algebraic groups have also a Lie group structure.

Example 1.5.

1. The previous example in 1.2 are algebraic groups, called linear algebraic groups.

2. Elliptic curves.

3. The group $\mathrm{PGL}(n,\mathbb{R})$ is a real algebraic group for $n \geq 1$, it can be seen as the group of automorphisms of the $n \times n$ matrices which is an algebraic subgroup of $\mathrm{Gl}(n^2,\mathbb{R})$. The same is valid for the complex case.

4. For n odd, $\mathrm{PSL}(n,\mathbb{R}) = \mathrm{PGL}(n,\mathbb{R})$ so it is algebraic. For n even, $\mathrm{PSL}(n,\mathbb{R}) = \mathrm{PGL}(n,\mathbb{R})^\circ$ which is a semialgebraic group (defined by polynomials inequalities).

A reductive connected algebraic group has another very useful characterization.

Proposition 1.6. *An algebraic group contains a unique maximal normal connected solvable subgroup called the radical.*

A connected complex algebraic group is reductive if and only if its radical is isomoprphic to $(\mathbb{C}^*)^n$*, for some* $n \geq 0$*. And it is semisimple if and only if its radical is trivial.*

Example 1.7. The algebraic group $\mathrm{Sl}(2,\mathbb{C})$ is a semisimple algebraic group with complex dimension 3. It is a non-compact, simple complex Lie group and irreducible. Its center is $Z(\mathrm{Sl}(2,\mathbb{C})) = \{\pm I_2\}$.

1.3 Finitely generated groups

A finitely generated group Γ is a group that has some finite generating set S such that every element of Γ can be written as the combination (under the group operation) of finitely many elements of S and of inverses of elements S. Most of the applications will be with Γ a finitely presented group. The finitely generated groups will be endowed with the discrete topology.

Example 1.8.

1. Finite groups, Integer group
2. Symmetric group, Dihedral group, Cyclic group
3. Free group

Surface groups

These are other examples of finitely generated groups which are fundamental groups of oriented surfaces.

Let $g \geq 0$ and $n \geq 0$ be integers. A group is called a *surface group* if it is isomorphic to

$$\pi_{g,n} := \left\langle a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n \ : \ \prod_{i=1}^{g}[a_i, b_i] = \prod_{j=1}^{n} c_j \right\rangle$$

with $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ the commutator of a_i and b_i. The case $n = 0$, where $\prod_{i=1}^{g}[a_i, b_i] = 1$, is called a *closed surface group.*

For $n \geq 1$, $\pi_{g,n}$ is isomorphic to the free group with $2g + n - 1$ generators. The surface groups are abelian in a very few cases. For example, $\pi_{g,0}$ is non-abelian for $g \geq 2$.

The following result gives the reason for the name of these groups.

Theorem 1.9. *Let* $\Sigma_{g,n}$ *be a connected orientable topological surface of genus* $g \geq 0$*, with* $n \geq 0$ *punctures. The fundamental group of* $\Sigma_{g,n}$*,* $\pi_1(\Sigma_{g,n})$*, is isomorphic to* $\pi_{g,n}$*.*

Proof. For the case $n = 0$ see Theorem 2.3.15 of [Lab13]. For the general case, a sphere with $n \geq 1$ punctures is homotopy equivalent to the wedge of $n-1$ circles. So, its fundamental group is the free group with $n - 1$ generators. Now, a surface of genus g with one puncture is homotopy equivalent to the wedge of $2g$ circles. Thus, its fundamental group is the free group with $2g$ generators. On the other hand, $\Sigma_{g,n}$ is the union of $\Sigma_{g,1}$ and $\Sigma_{0,n+1}$. To achieve the result apply Van Kampen Theorem (see for instance [Hat02]). □

2 Representation Varieties

Let Γ be a finitely generated group, and let G be a (real or complex) Lie group. The object studied in this section consists of the set of homomorphisms from Γ to G. More concretely,

Definition 2.1. For a finitely generated group Γ and a Lie group G, the *representation variety* associated to Γ and G, denoted by $\mathrm{Hom}(\Gamma, G)$, is the set

$$\mathrm{Hom}(\Gamma, G) = \{\rho : \Gamma \to G : \ \rho \text{ is a group homomorphism}\}$$

Topology

The representation variety $\mathrm{Hom}(\Gamma, G)$ can be endowed with the subspace topology induced by the compact-open topology in the space of continuous functions $\Gamma \to G$, where Γ is equipped with the discrete topology.

The sets

$$V(K, U) := \{f : \Gamma \to G : \ K \subset \Gamma \,\text{finite}, \, U \subset G \,\text{open}, \ f(K) \subset U\}$$

form a sub-basis for the compact-open topology on $\mathrm{Hom}(\Gamma, G)$.

Fixing the generators $\{\gamma_1, \cdots, \gamma_n\}$ of Γ, it can be considered the following subspace of G^n:

$$X(\Gamma, G) = \{(\rho(\gamma_1), \cdots, \rho(\gamma_n)) : \ \rho \in \mathrm{Hom}(\Gamma, G)\} \subset G^n$$

Proposition 2.2. *If G is a Lie group endowed with an analytic atlas, then $X(\Gamma, G)$ is an analytic subvariety of G^n homeomorphic to* $\mathrm{Hom}(\Gamma, G)$*. In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural analytic variety structure and the structure does not depend on the choice of generators of* Γ*.*

Proof. Consider a set of relations $\{r_i\}$, which can be infinite for the generators $\gamma_1, \cdots, \gamma_n$. The relations are defined by multiplications and inverse operations on G, which are analytic, thus they are also analytic maps $r_i : G^n \to G$. Now, $X(\Gamma, G)$ will be an analytic subset of G^n since the previous relations will induce relations of the form $r_i(g_1, \cdots, g_n) = 1$, for $g_i \in G$, in G^n. It is easy to see that a group homomorphism $\rho : \Gamma \to G$ can be determined by the image of a set of generators of Γ. From this idea, there is a bijection, Ψ, between $\mathrm{Hom}(\Gamma, G)$ and $X(\Gamma, G)$:

$$\begin{aligned} \Psi : \mathrm{Hom}(\Gamma, G) &\longrightarrow X(\Gamma, G) \\ \rho &\longmapsto (\rho(\gamma_1), \cdots, \rho(\gamma_n)) \end{aligned}$$

This bijection is a homeomorphism, as will be seen next. Pick a collection of open sets $U_1, \cdots, U_n$ of G, then

$$\Psi^{-1}(X(\Gamma, G) \cap (U_1 \times \cdots \times U_n)) = \mathrm{Hom}(\Gamma, G) \cap \bigcap_{i=1}^{n} V(\{\gamma_i\}, U_i)$$

Now, any element $\gamma \in \Gamma$ written in terms of generators will determine an analytic function $\Gamma : G^n \to G$ (recall that Ψ is a bijection). Let $K \subset \Gamma$ be a finite set and $U \subset G$ an open set, so

$$\Psi(\mathrm{Hom}(\Gamma, G) \cap V(K, U)) = X(\Gamma, G) \cap \bigcap_{k \in K} \gamma^{-1}(U)$$

Concluding that Ψ and its inverse are continuous, hence it is a homeomorphism.

Choosing another set of generators for Γ, $\{\gamma'_1, \cdots, \gamma'_n\}$, and denoting by $X'(\Gamma, G)$ the analogous to $X(\Gamma, G)$ for this new set, it can be defined a map between these two as the following composition

$$X(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) \to X'(\Gamma, G)$$

which is an isomorphism of analytic varieties. Indeed, as γ'_i is a word in the generators $\gamma_1, \cdots, \gamma_n$, thus $\rho(\gamma'_i)$ is a word in $\rho(\gamma_1), \cdots, \rho(\gamma_n)$. Thus, it is analytic as word maps are so. □

With a similar proof with minor adaptations, we get for G a real or complex algebraic group

Proposition 2.3. *If G is a real or complex algebraic group, then $X(\Gamma, G)$ is an algebraic subvariety of G^n. In particular, $\mathrm{Hom}(\Gamma, G)$ has a natural structure of real or complex algebraic variety and the structure does not depend on the choice of generators of Γ.*

Remark 2.4.

1. The group Γ can be defined by an infinite number of relations, thus the number of equations defining $X(\Gamma, G)$ may also be infinite. Using Hilbert's basis theorem, we know that any algebraic variety over a field can be described as the zero locus of a finitely number of polynomial equations.

2. If G is a real or complex algebraic group then it is also a Lie group. This implies that the representation variety $\mathrm{Hom}(\Gamma, G)$ has the two structures, of an analytic variety and an algebraic variety. For the analytic structure, the topology is called the standard topology and for the algebraic, the Zariski topology. The first one is always Hausdorff, the second one is coarser than the standard topology (the Zariski open sets are open in the standard topology, because the polynomials are continuous functions). A nonempty Zariski open set is dense in both topologies.

2.1 Symmetries: conjugation action of G.

In this section, we will describe symmetries of the representation variety $\mathrm{Hom}(\Gamma, G)$. Two natural ones are obtained by the right action of the automorphism group of Γ, $\mathrm{Aut}(\Gamma)$, by pre-composing and the left action of the automorphism group of G, $\mathrm{Aut}(G)$, by post-composing.

These actions preserve the analytic or algebraic structure of $\mathrm{Hom}(\Gamma, G)$, this is a consequence of the following more general property.

Proposition 2.5. *Let Γ, Γ_1 and Γ_2 be finitely generated groups and G, G_1 and G_2 be a Lie or algebraic groups.*

1. *If $f : \Gamma_1 \to \Gamma_2$ is a morphism, then the induced map $f^* : \mathrm{Hom}(\Gamma_1, G) \to \mathrm{Hom}(\Gamma_2, G)$ is an analytic or regular map.*

2. *If $g : G_1 \to \Gamma_2$ is a morphism, then the induced map $g_* : \mathrm{Hom}(\Gamma, G_1) \to \mathrm{Hom}(\Gamma, G_2)$ is an analytic or regular map.*

There is an important normal subgroup of $\mathrm{Aut}(G)$, whose elements are called inner automorphisms of G defined by conjugating by an element of G, denoted by $\mathrm{Inn}(G)$. More concretely, an inner automorphism of G has the following form: let $\rho \in \mathrm{Hom}(\Gamma, G)$ and $g \in G$, $(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$, for every $\gamma \in \Gamma$. As conjugating by elements of $Z(G)$ gives a trivial action, $\mathrm{Inn}(G) \cong G/Z(G)$.

In the case G is semisimple, $\mathrm{Inn}(G)$ is a finite index subgroup of $\mathrm{Aut}(G)$. Indeed, assume first that G is also simply connected, so the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ induced by derivation is an isomorphism of Lie groups. Because of this, it can be proved if we consider instead Lie algebras. If $\mathfrak{g}$ is semisimple, then the Lie algebras of $\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$ are isomorphic. Thus, $\mathrm{Inn}(\mathfrak{g})$ is a finite index subgroup of $\mathrm{Aut}(\mathfrak{g})$, consequently the same is true for groups. If G is not simply connected, consider the universal cover of G, using lifting properties we can conclude the result.

The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is important in many cases, for example the holonomy representations associated to structures on surfaces that are defined up to conjugation by an element of G. So, it worths studying the quotient

$$\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$$

This is a first idea of what will be a character variety associated to Γ and G.

2.2 Tangent spaces

In this section, we will study the tangent spaces of the representation varieties. Notice that these ones are not smooth in general. First, we must fix the notion of tangent space suitable for this case. It will be more adequate one that does not depend on choosing generators of Γ. In order to achieve such definition we use ringed spaces concept ([Kar92; Law09]).

Definition 2.6. A real valued ringed space is a toplogical space X with a sheaf $\mathcal{O}$ of real valued functions called admissible functions.

For $x \in X$, let $\mathcal{M}_x$ denote the germs of admissible functions at x that vanish at x.

The Zariski tangent space to X at $x \in X$ is the vector space

$$(\mathcal{M}_x/\mathcal{M}_x^2)^*$$

This definition generalizes the notion of tangent spaces for manifolds and of Zariski tangent spaces for analytic and algebraic varieties.

Example 2.7.

1. Smooth manifolds with the sheaf of smooth real valued functions
2. Analytic varieties with the sheaf of analytic functions
3. Algebraic varieties with the sheaf of regular maps

We will consider $\mathrm{Hom}(\Gamma, G)$ as a submanifold of the infinite product G^Γ, in this way the embedding does not depend on fixing generators of Γ. A function $G^\Gamma \to \mathbb{R}$ is said to be locally smooth if it is locally a smooth function of a finite number of coordinates. Define the sheaf $\mathcal{C}^\infty(G^\Gamma)$ of locally smooth functions on G^Γ as the direct limit of the sheaves $\mathcal{C}^\infty(G^I)$ of smooth functions on the manifold G^I, with $I \subset \Gamma$ a finite subset ($I \subset J \subset \Gamma$ finite subsets, $\mathcal{C}^\infty(G^I) \hookrightarrow \mathcal{C}^\infty(G^J)$). In this way, G^Γ with the sheaf $\mathcal{C}^\infty(G^\Gamma)$ is a real valued ringed space.

Proposition 2.8. *The Zariski tangent space to G^Γ at any point is isomorphic to $\mathfrak{g}^\Gamma$.*

Proof. The Zariski tangent space to G^Γ at a point f, $T_f G^\Gamma$, can be characterized as the vector space of tangent vectors to smooth deformations of f. Let $\exp : \mathfrak{g} \to G$ be the Lie exponential map. Consider $u \in \mathfrak{g}^\Gamma$ and the 1-parameter family of maps

$$\exp(tu)f$$

which defines a deformation of f. Can be shown that this map is an isomorphism between $T_f G^\Gamma$ and $\mathfrak{g}^\Gamma$. $\square$

The representation variety can be defined by the equations

$$F_{\alpha,\beta}(f) := f(\alpha\beta)f(\beta)^{-1}f(\alpha)^{-1} = 1, \quad \forall \alpha, \beta \in \Gamma$$

The sheaf of smooth functions on $\mathrm{Hom}(\Gamma, G)$ is defined as follows: let $U \subset G^\Gamma$ be an open subset, consider the open subset $\mathrm{Hom}(\Gamma, G) \cap U$ of $\mathrm{Hom}(\Gamma, G)$ and assign to it the quotient ring

$$C^\infty(U)/(\phi \circ F_{\alpha,\beta} : \alpha, \beta \in \Gamma)$$

with $\phi : G \to \mathbb{R}$ a smooth function such that $\phi(1) = 0$. So, $\mathrm{Hom}(\Gamma, G)$ can be equipped with a ringed space structure.

Let us see, in the next proposition, that choosing generators for Γ will induce the same structure

Proposition 2.9. *If a set of n generators is fixed for Γ and F_n is the free group with n generators, then the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, G) & \longrightarrow & G^n \\ \downarrow & & \downarrow \\ G^\Gamma & \longrightarrow & G^{F_n} \end{array}$$

The inclusion $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$ of ringed spaces induces an inclusion of the Zariski tangent spaces $T_\rho \mathrm{Hom}(\Gamma, G) \subset \mathfrak{g}^\Gamma$, for $\rho \in \mathrm{Hom}(\Gamma, G)$. The space $T_\rho \mathrm{Hom}(\Gamma, G)$ is the intersection of the kernels of the linear forms $D_\rho F_{\alpha,\beta} : \mathfrak{g}^\Gamma \to \mathfrak{g}$, for all $\alpha, \beta \in \Gamma$. By definition, for $v \in \mathfrak{g}^\Gamma$ and $\rho \in \mathrm{Hom}(\Gamma, G)$,

$$\begin{aligned} D_\rho F_{\alpha,\beta}(v) &= \left.\frac{d}{dt}\right|_{t=0} \phi_{\alpha,\beta}(\exp(tv)\rho) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(tv(\alpha\beta))\rho(\alpha\beta)\rho(\beta)^{-1}\exp(-tv(\beta))\rho(\alpha)^{-1}\exp(-tv(\alpha)) \\ &= v(\alpha\beta) - v(\alpha) - \mathrm{Ad}(\rho(\alpha))v(\beta) \end{aligned}$$

From this computation, the Zariski tangent space of $\mathrm{Hom}(\Gamma, G)$ at a point ρ can be characterized

$$T_\rho \mathrm{Hom}(\Gamma, G) = \{v \in \mathfrak{g}^\Gamma : v(\alpha\beta) = v(\alpha) + \mathrm{Ad}(\rho(\alpha))v(\beta), \quad \forall \alpha, \beta \in \Gamma\}$$

Tangent space as cocycles

Another characterization of the tangent space can be given in terms of group cohomology. For that, an element $\rho \in \mathrm{Hom}(\Gamma, G)$ endows $\mathfrak{g}$ with a Γ-module structure defined by

$$\Gamma \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

The Lie algebra $\mathfrak{g}$ seen as a Γ-module will be denoted by $\mathfrak{g}_\rho$.

Now, consider the cochain complex given by

$$C^k(\Gamma, \mathfrak{g}_\rho) := \mathrm{Map}(\Gamma^k, \mathfrak{g}_\rho), \quad k \geq 0$$

where $\mathrm{Map}(\Gamma^k, \mathfrak{g}_\rho)$ is the Γ-module of set-theoretic maps from Γ^k to $\mathfrak{g}_\rho$. The differential map $\partial^k : C^{k-1}(\Gamma, \mathfrak{g}_\rho) \to C^k(\Gamma, \mathfrak{g}_\rho)$ is defined as follows, for $u \in \mathrm{Map}(\Gamma^k, \mathfrak{g}_\rho)$ and $(\gamma_1, \cdots, \gamma_k) \in \Gamma^k$

$$
\begin{aligned}
(\partial^k u)(\gamma_1, \cdots, \gamma_k) \ &:= \ \gamma_1 \cdot u(\gamma_2, \cdots, \gamma_k) + \\
&+ \ \sum_{i=1}^{k-1} (-1)^i u(\gamma_1, \cdots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \cdots, \gamma_k) + (-1)^k u(\gamma_1, \cdots, \gamma_{k-1})
\end{aligned}
$$

It is true that $\partial^k \partial^{k-1} = 0$, for every $k \geq 1$. Denote by $Z^k(\Gamma, \mathfrak{g}_\rho)$ and $B^k(\Gamma, \mathfrak{g}_\rho)$ the sets of k-cocycles and of k-coboundaries of the complex, respectively. The space of 1-cocycles is

$$Z^1(\Gamma, \mathfrak{g}_\rho) = \{v \in \mathfrak{g}^\Gamma : v(\gamma_1\gamma_2) = v(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1))v(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma\}$$

which is equal to the tangent space $T_\rho \mathrm{Hom}(\Gamma, G)$.

The space of 1-coboundaries is

$$B^1(\Gamma, \mathfrak{g}_\rho) = \{v \in \mathfrak{g}^\Gamma : \exists \zeta \in \mathfrak{g}, \, v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta, \quad \forall \gamma \in \Gamma\}$$

2.3 Smooth points

A point x of an analytic vatiety $X \subset \mathbb{R}^n$ is said to be a *smooth point* if there exists an open neighbourhood $U \subset X$ of x such that U is an embedded submanifold of $\mathbb{R}^n$.

We can use the Implicit Function Theorem, and characterize the smooth points as being the points where the rank of the Jacobian is maximal. This is equivalent to the dimension of the Zariski Tangent space of X at a smooth point being minimal. If every point of an analytic variety is smooth, then it is an analytic manifold.

Representation varieties are analytic varieties. If Γ is a free group, then $\mathrm{Hom}(\Gamma, G)$ is an analytic manifold because there is no relations in its definition.

The following property is satisfied

Proposition 2.10. *The set of smooth points of* $\mathrm{Hom}(\Gamma, G)$ *is invariant by the conjugation action of* G*.*

Proof. The conjugation action of G is analytic. Thus, it maintains smooth neighbourhoods of the points contained in $\mathrm{Hom}(\Gamma, G)$. Other way is to notice that the Zariski tangent spaces at ρ and $g\rho g^{-1}$ are isomorphic as Γ-modules ($v \in Z^1(\Gamma, \mathfrak{g}_\rho) \mapsto \mathrm{Ad}(g)v \in Z^1(\Gamma, \mathfrak{g}_{g\rho g^{-1}})$), then they have the same dimension. □

For the case where where $\Gamma = \pi_{g,0}$ is a closed surface group and G is a reductive Lie group, it is possible to characterize the smooth points explicitly (see [Gol84]). Denote by $Z(\rho)$ the centralizer of $\rho(\Gamma)$ inside G, which is called the *stabilizer* of ρ for the conjugation action.

Theorem 2.11. *Let G be a reductive Lie group, then the dimension of the Zariski tangent space to* $\mathrm{Hom}(\pi_{g,0}, G)$ *at ρ is*

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\rho) = (2g-1)\dim G + \dim Z(\rho)$$

The representations ρ such that $\dim Z(\rho) = \dim Z(G)$ *minimize the dimension of their Zariski tangent space.*

Proof. To compute the dimension of the of the Zariski tangent space to $\mathrm{Hom}(\pi_{g,0}, G)$ at ρ, we use its identification with $Z^1(\pi_{g,0}, \mathfrak{g}_\rho)$. The group cohomology of $\pi_{g,0}$ with coefficients in $\mathfrak{g}_\rho$ is isomorphic to the de Rham cohomology of the surface $\Sigma_{g,0}$ with coefficients in the adjoint bundle of the principal bundle G-bundle associate to ρ (see section 4. or [Gol84]). Thus, it vanishes in degrees larger than 2.

It can be proved that the Euler characteristic

$$\dim H^0(\pi_{g,0}, \mathfrak{g}_\rho) - \dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\rho)$$

does not depend on ρ, because it can be expressed as the alternating sum of the dimensions of the spaces of cochains. These ones in simplicial cohomology with local coefficients have finite dimension and are independent of ρ as the structure of the $\pi_{g,0}$-module of $\mathfrak{g}_\rho)$ it is only used in the differential.

So, if we choose ρ as the trivial representation, then $\mathfrak{g}_\rho)$ is a trivial $\pi_{g,0}$ module and the previous Euler characteristic will be equal to the Euler characteristic of $\Sigma_{g,0}$ multiplied by $\dim G$. Thus,

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) = (2g-2)\dim G + \dim H^0(\pi_{g,0}, \mathfrak{g}_\rho) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\rho)$$

By the Poincaré duality,

$$H^2(\pi_{g,0}, \mathfrak{g}_\rho) \simeq H^0(\pi_{g,0}, \mathfrak{g}_\rho^*)^*$$

As G is reductive, there exists a non-degenerate, Ad-invariant, bilinear form on $\mathfrak{g}$, this gives an $\pi_{g,0}$-modules isomorphism $\mathfrak{g}_\rho) \simeq \mathfrak{g}_\rho)^*$. Then, $\dim H^2(\pi_{g,0}, \mathfrak{g}_\rho) = \dim H^0(\pi_{g,0}, \mathfrak{g}_\rho)$. On the other hand, $H^0(\pi_{g,0}, \mathfrak{g}_\rho) = \mathfrak{z}(\rho)$. Hence,

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) = (2g-2)\dim G + 2\dim Z(\rho)$$

We know $\dim B^1(\pi_{g,0}, \mathfrak{g}_\rho) = \dim G - \dim Z(\rho)$. Concluding that

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\rho) = (2g-1)\dim G + \dim Z(\rho)$$

By the inclusion $Z(G) \subset Z(\rho)$, we get $\dim Z(G) \leq \dim Z(\rho)$, then ρ minimizes the dimension of its Zariski tangent space if and only if $\dim Z(G) = \dim Z(\rho)$. ◻

2.4 Characterization of orbits

Consider the action of $\mathrm{Inn}(G) \cong G/Z(G)$ on $\mathrm{Hom}(\Gamma, G)$, for an element $\rho \in \mathrm{Hom}(\Gamma, G)$, denote by $\mathcal{O}_\rho$ the orbit of ρ by this action. Denote by $Z(\rho)$ the centralizer of $\rho(\Gamma)$ inside G, which is called the stabilizer of ρ for the conjugation action. This is a closed subgroup of G.

The orbit $\mathcal{O}_\rho$ is a smooth manifold isomorphic to the quotient $G/\mathrm{Z}(\rho)$. A smooth deformation of ρ inside $\mathcal{O}_\rho$ has the form $\rho_t = g(t)\rho g(t)^{-1}$, with $g(t)$ a smooth 1-parameter family inside G such that $g(0) = 1$. The tangent vector to ρ_t at $t = 0$ is the coboundary

$$v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta$$

for every $\gamma \in \Gamma$ and where $\zeta \in \mathfrak{g}$ is a tangent vector to $g(t)$ at $t = 0$.

Conversely, if $\zeta \in \mathfrak{g}$, the coboundary $v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta$ is tangent to $\exp(t\zeta)\rho\exp(-t\zeta)$ at $t = 0$.

Thus, we can conclude that

$$T_\rho\mathcal{O}_\rho = B^1(\Gamma, \mathfrak{g}_\rho)$$

The following inclusions of ringed spaces

$$\mathcal{O}_\rho \subset \operatorname{Hom}(\Gamma, G) \subset G^\Gamma$$

induces a chain of inclusions on Zariski tangent spaces

$$\begin{array}{ccccc} T_\rho O_\rho & \hookrightarrow & T_\rho \operatorname{Hom}(\Gamma, G) & \hookrightarrow & T_\rho G^\Gamma \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\ B^1(\Gamma, \mathfrak{g}_\rho) & \hookrightarrow & Z^1(\Gamma, \mathfrak{g}_\rho) & \hookrightarrow & C^1(\Gamma, \mathfrak{g}_\rho) \end{array}$$

Denoting by $\mathfrak{z}(\rho)$ the Lie algebra of $Z(\rho)$, the space of 1-coboundaries, $B^1(\Gamma, \mathfrak{g}_\rho)$, can be identified with the quotient $\mathfrak{g}/\mathfrak{z}(\rho)$. So, to compute the dimensions it can be used the following equality

$$\dim B^1(\Gamma, \mathfrak{g}_\rho) = \dim \mathcal{O}_\rho = \dim G - \dim Z(\rho)$$

It is easy to see that the conjugation action of $\operatorname{Inn}(G) \cong G/Z(G)$ on $\operatorname{Hom}(\Gamma, G)$ is not free, notice that the trivial representation is a fixed point. Other easy thing is that for $\rho \in \operatorname{Hom}(\Gamma, G)$, $Z(G) \subset Z(\rho)$.

Consider the subset of $\operatorname{Hom}(\Gamma, G)$ defined by

$$\{\rho \in \operatorname{Hom}(\Gamma, G) : \ Z(\rho) = Z(G)\}$$

This set is invariant by the conjugation action and this action is free on it. Moreover,

Proposition 2.12. *The conjugation action on* $\operatorname{Hom}(\Gamma, G)$ *is locally free (i.e, the stabilizer of every element is discrete) if and only if* $\dim Z(G) = \dim Z(\rho)$*, for every* $\rho \in \operatorname{Hom}(\Gamma, G)$*.*

Proof. The conjugation action on $\operatorname{Hom}(\Gamma, G)$ induces a surjective linear map, for each $\rho \in \operatorname{Hom}(\Gamma, G)$, $\psi_\rho : \mathfrak{Inn}(G) \to T_\rho \mathcal{O}_\rho$, where $\mathfrak{Inn}(G)$ is the Lie algebra of $\operatorname{Inn}(G)$. This map is defined by
$\psi_\rho(\zeta) = \frac{d}{dt}\big|_{t=0} \exp(t\zeta)(\rho)$.

The conjugation action in $\operatorname{Hom}(\Gamma, G)$ is locally free in ρ if and only if ψ_ρ is injective. The last will be true, as ψ_ρ is surjective if and only if $\mathfrak{Inn}(G)$ and $T_\rho \mathcal{O}_\rho$ have the same dimension. The dimension of $\mathfrak{Inn}(G)$ is $\dim G - \dim Z(G)$ and of $T_\rho \mathcal{O}_\rho$ is $\dim G - \dim Z(\rho)$. Thus, the dimensions are equal if and only if $\dim Z(G) = \dim Z(\rho)$. □

In the case $\Gamma = \pi_{g,0}$ and G is a reductive Lie group, the smooth points of $\operatorname{Hom}(\pi_{g,0}, G)$ are exactly those for which the conjugation action is locally free.

2.5 Subsets of representations

The last proposition of the previous subsection motivates the following classes of representations.

Definition 2.13. A representation $\rho \in \operatorname{Hom}(\Gamma, G)$ is said to be *regular* if $\dim Z(G) = \dim Z(\rho)$ and is said to be *very regular* if $Z(G) = Z(\rho)$.

We denote by $\operatorname{Hom}^{reg}(\Gamma, G)$ the $\operatorname{Inn}(G)$-invariant subspace of regular representations and by $\operatorname{Hom}^{vreg}(\Gamma, G)$ the $\operatorname{Inn}(G)$-invariant subspace of very regular representations.

Example 2.14. The representations $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ that are not regular (exercise).

In order to know whether two orbits can be separated or not by disjoint open sets in the topological quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$, which is crucial for this quotient to have a nice structure, we introduce the notion of Borel and parabolic subgroups.

Definition 2.15. A *Borel* subgroup of a complex algebraic group G is a maximal, Zariski closed, solvable connected subgroup of G. A *parabolic* subgroup of a real or complex algebraic group G is a Zariski closed subgroup of G that contains a Borel subgroup over $\mathbb{C}$. A *Levi* subgroup of a real or complex algebraic group G is a connected subgroup isomorphic $G/R_u(G)$, where $R_u(G)$ is the unipotent subgroup of the radical of G.

A subgroup of an algebraic group is said to be irreducible if it is not contained in a proper parabolic subgroup of G.

Example 2.16. Let $G = \mathrm{Gl}(n, \mathbb{C})$. The Borel subgroups of G are those that preserve a full flag in $\mathbb{C}^n$. For example, the subgroup of upper triangular matrices is a Borel subgroup. The parabolic subgroups are those that preserve a partial flag in $\mathbb{C}^n$.

Definition 2.17. Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is said to be *irreducible* if the image of ρ is not included in a proper parabolic subgroup of G. We denote by $\mathrm{Hom}^{irr}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of irreducible representations.

Remark 2.18.

1. A representation $\rho : \Gamma \to G$ is irreducible if and only if its image is an irreducible subgroup of G.

2. It can happen that a representation is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$.

3. If $G = \mathrm{Gl}(n, \mathbb{C})$, then ρ is irreducible if snf only if $\mathbb{C}^n$ is an irreducible Γ-module. (Example 2.16)

4. If $G = \mathrm{Sl}(2, \mathbb{C})$, the irreducible representations can be characterized in terms of traces. In fact, a representation $\rho : \Gamma \to G$ can be shown to be irreducible if and only if there exists an element $\gamma \in [\Gamma, \Gamma] \subset \Gamma$, of the commutator of Γ such that $\mathrm{Tr}(\rho(\gamma)) \neq 2$. (Lamma 1.2.2 of [CS83])

For the case of a reductive algebraic group, the irreducible representations have further properties.

Proposition 2.19 ([Sik12], Propositions 27 and 28)**.** *Let G be a reductive algebraic group. Then*

1. $\mathrm{Hom}^{irr}(\Gamma, G) \subset \mathrm{Hom}^{reg}(\Gamma, G)$. *(the centralizer of an irreducible subgroup of G is a finite extension of $Z(G)$)*

2. $\mathrm{Hom}^{irr}(\Gamma, G)$ *is Zariski open in* $\mathrm{Hom}(\Gamma, G)$.

3. *If $\Gamma = \pi_{g,n}$ is a surface group, then* $\mathrm{Hom}^{irr}(\pi_{g,n}, G)$ *is dense in a nonempty set of irreducible components of* $\mathrm{Hom}(\pi_{g,n}, G)$.

The important result is the following.

Theorem 2.20 ([JM87]). *Let G be a reductive algebraic group. The conjugation action on* $\mathrm{Hom}^{irr}(\Gamma, G)$ *is proper.*

Representations that are irreducible and very regular are called *good* representations; this notion was introduced by Jonhson and Milson in [JM87]. We denote by $\mathrm{Hom}^{gd}(\Gamma, G)$ the conjugation invariant subspace of $\mathrm{Hom}(\Gamma, G)$ of good representations.

According to the previous results, the conjugation action on $\mathrm{Hom}^{g}(\Gamma, G)$ is free and proper.

Proposition 2.21. *Let G be a reductive algebraic group.*

1. $\mathrm{Hom}^{gd}(\Gamma, G)$ *is Zariski open in* $\mathrm{Hom}(\Gamma, G)$*.*
2. *If* $\Gamma = \pi_1 \Sigma_{g,0}$*,*
   - *(a)* $\mathrm{Hom}^{gd}(\pi_1 \Sigma_{g,0}, G)$ *is an analytic manifold of dimension* $(2g-1)\dim G + \dim Z(G)$*.*
   - *(b) The conjugation action on* $\mathrm{Hom}^{gd}(\pi_1 \Sigma_{g,0}, G)$ *is proper and free.*
   - *(c) The quotient* $\mathrm{Hom}^{gd}(\pi_1 \Sigma_{g,0}, G)/\mathrm{Inn}(G)$ *is an analytic manifold of dimension* $(2g-2)\dim G + 2\dim Z(G)$ *(always even).*

There is a generalization of irreducible representation, that is, of reductive (or completely reducible) representation.

In order to give this new subset of representations, we introduce the following definitions for an algebraic group.

Definition 2.22. 1. An algebraic group is called *linearly reductive* if all its finite-dimensional representations are completely reducible, i.e, are a sum of simple (the only subrepresentations are the trivial and itself) subrepresentations.

2. A subgroup of an algebraic group is called *completely reducible* if its Zariski closure is linearly reductive.

It can be shown that over the real or complex numbers, an algebraic group is linearly reductive if and only if its identity Zariski component is reductive (see Corollary 22.43 of [Mil17].

Now, we have the notion of a reductive representation.

Definition 2.23. Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is called *reductive* or *completely reducible* if $\rho(\Gamma) \subset G$ is completely reducible. The conjugation invariant subspace of reductive representations is denoted by $\mathrm{Hom}^{red}(\Gamma, G)$.

Example 2.24. A representation $\rho : \Gamma \to \mathrm{Gl}(n, \mathbb{C})$ is reductive if and only if $\mathbb{C}^n$ is a completely reducible Γ-module, i.e., a direct sum of irreducible Γ-modules. This is equivalent to have that every Γ-invariant subspace of $\mathbb{C}^n$ has a Γ-invariant complement.

Proposition 2.25. *Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is reductive if and only if $\rho(\Gamma)$ is contained in a parabolic subgroup P of G then it is contained in a Levi subgroup of P.*

For a reductive algebraic group, we have the following relations between these different types of representation.

Proposition 2.26. *Let G be a reductive algebraic group. Then,*

1. $\mathrm{Hom}^{irr}(\Gamma, G) \subset \mathrm{Hom}^{red}(\Gamma, G)$

2. $\mathrm{Hom}^{irr}(\Gamma, G) = \mathrm{Hom}^{red}(\Gamma, G) \cap \mathrm{Hom}^{reg}(\Gamma, G)$

Also, for the reductive algebraic group case, a reductive representation has another characterization.

Proposition 2.27. *Let G be a reductive algebraic group. A representation $\rho : \Gamma \to G$ is reductive if and only if the conjugation orbit $\mathcal{O}_\rho$ of ρ is closed in* $\mathrm{Hom}(\Gamma, G)$.

A consequence of this property is that the topological quotient

$$\mathrm{Hom}^{red}(\Gamma, G)/\mathrm{Inn}(G)$$

is a $\mathcal{T}_1$ space. Moreover, it can be proved that it is also Hausdorff (see [RS90]).

3 Character varieties

A first example of a Character variety is the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ endowed with the quotient topology. In this section, we will describe other ways to perform the quotient such that more interesting geometrical and topological properties will be satisfied by it.

As noted above, the conjugation action considered in the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ is, in general, not free and proper.

Recall that a topological space X is $\mathcal{T}_1$ if for any pair of different points of X, one is in an open set that does not contain the other point, this is equivalent to all the points of X being closed. A topological space X is $\mathcal{T}_2$ or *Hausdorff* if for any pair of different points in X, there are two disjoint open sets such that each contains one of the points.

One property that must be satisfied by a character variety is that there must be a projection from $\mathrm{Hom}(\Gamma, G)$ that factors through $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. So, we look for the largest possible finer quotient of $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ with a nice topology, regular properties, or that has a structure of a variety of a smooth manifold.

Some examples

1. Consider G an abelian algebraic group. In this case, the conjugation action is trivial. So,
$$\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G) = \mathrm{Hom}(\Gamma, G)$$
Any representation $\Gamma \to G$ factorizes through the abelianization $\Gamma^{ab} := \Gamma/[\Gamma, \Gamma]$ of Γ. Then, $\mathrm{Hom}(\Gamma, G) = \mathrm{Hom}(\Gamma^{ab}, G)$.

2. If $G = \mathbb{R}$ and $\Gamma = \pi_1 X$ is the fundamental group of some connected topological space X, there is a particular interpretation of $\mathrm{Hom}(\Gamma^{ab}, G)$. By Hurewicz Theorem, the abelianization of $\pi_1 X$ is isomorphic to the first cohomology group $H^1(X, \mathbb{R})$ of X. Thus,
$$\mathrm{Hom}(\pi_1(X), \mathbb{R}) = \mathrm{Hom}(H_1(X, \mathbb{R}), \mathbb{R}) = H^1(X, \mathbb{R})$$
where $H^1(X, \mathbb{R})$ is the vector space given by the first cohomology of X.

3. Consider $\Gamma = \mathbb{Z}$, a free group with one generator. In this case, $\mathrm{Hom}(\mathbb{Z}, G)/\mathrm{Inn}(G)$ is the space of conjugacy classes of G. See example with $G = \mathrm{PSl}(2, \mathbb{R})$. This is not a Hausdorff topological space, not even $\mathcal{T}_1$.

3.1 Hausdorff and $\mathcal{T}_1$ quotient

First, we give a way to construct the largest Hausdorff quotient space. In order to achieve this, let us define an equivalence relation.

Consider X a topological space and all equivalence relations $\approx$ on X such that $X/\approx$ is Hausdorff (such a relation exists, for example identify all the points of X).

Define the following equivalence relation in X: $x \sim y$ if and only if $x \approx y$ for all $\approx$.

The quotient $\mathrm{Haus}(X) := X/\sim$ is called the Hausdorffization of X. This space is a Hausdorff topological space and has the following universal property: if Y is a Hausdorff topological space, then any continuous map $X \to Y$ factors uniquely through the projection $X \to \mathrm{Haus}(X)$.

Denote by $[x]$ the equivalence class defined by the relation $\sim$.

Proposition 3.1. *Let $x, y \in X$ such that $\overline{[x]} \cap \overline{[y]} \neq \emptyset$ then $x \sim y$.*

Proof. As $\mathrm{Haus}(X)$ is Hausdorff then its points are closed. Then, the equivalence classes for $\sim$ are closed subsets of X. If $x \not\sim y$, then the equivalence classes of x and y are closed disjoint subsets of X. □

The Hausdorff character variety of a finitely generated group Γ in a Lie group G is defined as

$$\mathrm{Rep}^{Haus}(\Gamma, G) := \mathrm{Haus}\left(\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)\right)$$

See reference [Mon16] where this quotient is used.

In [RS90], a $\mathcal{T}_1$ quotient is defined. For a topological group G acting on a space X, denote the G-orbit of $x \in X$ by $\mathcal{O}_x$. We introduce the following assumption:

$$\text{For every } x \in X,\ \overline{\mathcal{O}_x} \subset X \text{ contains a unique closed } G\text{-orbit.} \tag{3.2}$$

We denote the set of closed orbits for the action of G on X by $X/\!/G$ and define the map $\pi : X \to X/\!/G$ which sends x to the unique closed orbit contained in $\overline{\mathcal{O}_x}$. The topology on $X/\!/G$ is defined, such that π is a quotient map, that is, $Z \subset X/\!/G$ is closed if and only if $\pi^{-1}(Z) \subset X$ is closed. This is equivalent to consider the next equivalence relation on X

$$x \sim y \quad \text{if and only if} \quad \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \emptyset$$

Proposition 3.3.

1. *$X/\!/G$ is homeomorphic to $X/\sim$.*

2. *$X/\!/G$ satisfies the following universal property: for every $\mathcal{T}_1$ space Y, any continuous map $X \to Y$ that is constant on G-orbits factors uniquely through $\pi : X \to X/\!/G$.*

3. *There exists a natural surjective continuous map $X/\!/G \twoheadrightarrow \mathrm{Haus}(X/G)$ such that*

$$\begin{array}{ccc} X & \twoheadrightarrow & X/G \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X/\!/G & \twoheadrightarrow & \mathrm{Haus}(X/G) \end{array}$$

   commutes

If $X//G$ is $\mathcal{T}_1$, then it will be the largest $\mathcal{T}_1$ quotient of X. If $X//G$ is Hausdorff then it is homeomorphic to $\mathrm{Haus}(X)$.

Returning to the G-conjugation action on $\mathrm{Hom}(\Gamma, G)$ and assuming property 3.2, the $\mathcal{T}_1$ character variety of Γ in G is defined to be

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)//\mathrm{Inn}(G)$$

It does not have to be a $\mathcal{T}_1$ space, but it always lies over any $\mathcal{T}_1$ quotient of $\mathrm{Hom}(\Gamma, G)$ (Proposition 3.3). Also, there is a surjection

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \twoheadrightarrow \mathrm{Rep}^{\mathrm{Haus}}(\Gamma, G)$$

which is a homeomorphism if $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ is Hausdorff.

3.2 Algebraic quotient

This quotient is based on the structure of the group G, using the so called geometric invariant theory (GIT). Its construction starts from an algebra of invariant regular functions. Some references to this subsection are [Dre04], [Lou15] and [Sik12], where more details can be found.

We begin by introducing some notions to describe this construction.

Definition 3.4. A function $\mathrm{Hom}(\Gamma, G) \to \mathbb{K}$, where $\mathbb{K}$ is a field is said to be an invariant function if it is invariant under conjugation by G.

For us $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The set $\mathrm{Hom}(\Gamma, G)$ has a structure of an algebraic variety. The algebra of regular functions on $\mathrm{Hom}(\Gamma, G)$, also called its coordinate ring, is denoted by $\mathbb{K}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of invariant regular functions by $\mathbb{K}[\mathrm{Hom}(\Gamma, G)]^G$.

The goal is to find a collection of invariant regular functions such that, any invariant regular function can be written as a polynomial expression in those generating functions.

To construct invariant functions $\mathrm{Hom}(\Gamma, G) \to \mathbb{K}$ from a conjugation action invariant function $f : G \to \mathbb{K}$, choose an element $\gamma \in \Gamma$ and define the function $f_\gamma : \mathrm{Hom}(\Gamma, G) \to \mathbb{K}$ by $f_\gamma(\rho) := f(\rho(\gamma))$. Consider the case when G is a linear algebraic group, in this case G is isomorphic to a closed subgroup of $\mathrm{Gl}(m, \mathbb{K})$. Examples of conjugation invariant functions include the trace function and the determinant function. The subalgebra of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ generated by trace functions is denoted by $\mathcal{T}(\Gamma, G)$.

Example 3.5. Fricke and Vogt in [FV97] proved that $\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{Sl}(2, \mathbb{C}))^{\mathrm{Sl}(2,\mathbb{C})}]$ is (linearly) generated by trace functions Tr_γ, for $\gamma \in \Gamma$. A trace function $\mathrm{Tr}_\gamma : \mathrm{Hom}(\Gamma, \mathrm{Sl}(2, \mathbb{C})) \to \mathbb{C}$ is an invariant function defined by $\mathrm{Tr}_\gamma(\rho) := \mathrm{Tr}(\rho(\gamma))$. In $\mathrm{Sl}(2, \mathbb{C})$, the trace relation

$$\mathrm{Tr}_{\gamma_1\gamma_2} + \mathrm{Tr}_{\gamma_1^{-1}\gamma_2} = \mathrm{Tr}_{\gamma_1} + \mathrm{Tr}_{\gamma_2}$$

for γ_1, $\gamma_2 \in \Gamma$.

Consider $\Gamma = F_n$ a free group with n generators, in this case, we have $\mathrm{Hom}(F_n, \mathrm{M}_m(\mathbb{K})) \cong \mathrm{M}_m(\mathbb{K})^n$. The function $\mathrm{M}_m(\mathbb{K})^2 \to \mathbb{K}$ defined by $(X, Y) \to \mathrm{Tr}(XY^\top)$ is invariant under $\mathrm{SO}(m, \mathbb{K})$-conjugation but not under $\mathrm{Sl}(m, \mathbb{K})$-conjugation.

Procesi in [Pro76], could prove in several cases of G, that $\mathbb{K}[\mathrm{M}_m(\mathbb{K})^n]^G$ is finitely generated by trace polynomials.

And if $\mathrm{M}_m(\mathbb{K})$ is replaced by another linear algebraic group? It will depend on the chosen group. For example, the function $\det^{-1} : \mathrm{Gl}(m, \mathbb{C}) \to \mathbb{C}$ is non-trivial and invariant under $\mathrm{Gl}(m, \mathbb{C})$-conjugation, but its restriction to $\mathrm{Sl}(m, \mathbb{C})$ is the constant function 1. By the Cayley-Hamilton theorem, the inverse of the determinant can be expressed as a rational function of traces. It can be shown that $\mathbb{C}[\mathrm{Hom}(F_n, \mathrm{Gl}(m, \mathbb{K}))^{\mathrm{Gl}(m,\mathbb{K})}]$ is generated by Tr and $\det^{-1}$, for $\gamma \in F_n$; $\mathbb{C}[\mathrm{Hom}(F_n, \mathrm{Sl}(m, \mathbb{K}))^{\mathrm{Sl}(m,\mathbb{K})}]$ is generated by Tr, for $\gamma \in F_n$ (see this and other cases in the notes [Mar22]).

Let now Γ be a finitely generated group with generators $\{\gamma_1, \cdots, \gamma_n\}$. The embedding

$$\iota : \mathrm{Hom}(\Gamma, G) \hookrightarrow G^n = \mathrm{Hom}(F_n, G)$$

induces a surjective morphism

$$\iota^* : \mathbb{C}[G^n] \twoheadrightarrow \mathbb{C}[\mathrm{Hom}(\Gamma, G)]$$

This morphism maps invariant functions to invariant functions, thus it restricts to the morphism

$$(\iota^*)^G : \mathbb{C}[G^n]^G \twoheadrightarrow \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$$

If G is a reductive linear algebraic group, then $(\iota^*)^G$ is surjective, because of the existence of the so-called Reynolds operators ([Sik13]). Moreover, if the algebra $\mathbb{C}[G^n]^G$ is generated by trace functions, then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is equal to the subalgebra of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ generated by trace functions, that is, $\mathcal{T}(\Gamma, G)$. For example, this is true for $G = \mathrm{Sl}(m, \mathbb{C})$.

An important result is the Nagata Theorem that says in the case G is a reductive algebraic group over $\mathbb{C}$, then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated (see, for instance, [Dol03], Theorem 3.3). In this case, there is an algebraic variety called the spectrum of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ and denoted by

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) := \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$$

whose algebra of regular function is $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. The points of this variety belong to the image of $\mathrm{Hom}(\Gamma, G)$ in a family of generators of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. It is called the *GIT character variety* of Γ in the reductive group G and it is also denoted by $\mathrm{Hom}(\Gamma, G)//G$.

The GIT character variety has the structure of a complex algebraic variety, it is a Hausdorff space for the Euclidean topology. The inclusion $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ induces a surjective morphism

$$p : \mathrm{Hom}(\Gamma, G) \twoheadrightarrow \mathrm{Hom}(\Gamma, G)//G$$

Next, we give some properties of this quotient.

Proposition 3.6.

1. *(Universal property) For every algebraic variety Y, any morphism $\mathrm{Hom}(\Gamma, G) \to Y$ that is constant on G-orbits factors uniquely through p.*

2. *For any representations ρ_1, $\rho_2 \in \mathrm{Hom}(\Gamma, G)$, $p(\rho_1) = p(\rho_2)$ if and only if $\overline{\mathcal{O}_{\rho_1}} \cap \overline{\mathcal{O}_{\rho_2}} \neq \emptyset$.*

3. *The fibers of p contain a unique closed orbit.*

From these properties, we can conclude that the GIT quotient coincides with the $\mathcal{T}_1$ character variety and also with the Hausdorff character variety, that is,

$$\mathrm{Hom}(\Gamma, G)//G \cong \mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \cong \mathrm{Rep}^{\mathrm{Haus}}(\Gamma, G)$$

Another way to characterize GIT character variety is in terms of the notion of stability on representations.

Definition 3.7. Let G be an algebraic group and Γ a finitely generated group. A representation $\rho : \Gamma \to G$ is

1. *polystable* if $\mathcal{O}_\rho$ is closed.

2. *stable* if it is polystable and regular.

We denote by $\mathrm{Hom}^{ps}(\Gamma, G)$ (resp. $\mathrm{Hom}^{s}(\Gamma, G)$) the conjugation invariant polystable (resp. stable) representations.

In the case of a reductive algebraic group, as has been seen previously, $\rho \in \mathrm{Hom}(\Gamma, G)$ is polystable if and only if it is reductive, and it is stable if and only if it is irreducible. Also, the following result is valid

Theorem 3.8. *Let G be a reductive complex algebraic group. Then we have the homeorphism between topological quotient*

$$\mathrm{Hom}^{ps}(\Gamma, G)/\mathrm{Inn}(G) = \mathrm{Hom}^{red}(\Gamma, G)/\mathrm{Inn}(G) \cong \mathrm{Hom}(\Gamma, G)//G$$

And $\mathrm{Hom}(\Gamma, G)//G$ *contains the Zariski open subset* $\mathrm{Hom}^{s}(\Gamma, G)/\mathrm{Inn}(G) = \mathrm{Hom}^{irr}(\Gamma, G)/\mathrm{Inn}(G)$.

Proof. Polystable representations have a closed orbit under the $\mathrm{Inn}(G)$-action by definition. Thus, the projection p factors through the injective map

$$\mathrm{Hom}^{ps}(\Gamma, G)/\mathrm{Inn}(G) \to \mathrm{Hom}(\Gamma, G)//G$$

This map is also surjective. □

3.3 Strong Deformation Retractions

In this subsection, G will always be a connected, reductive, linear algebraic group over $\mathbb{C}$. It can be seen as a closed subgroup of a $\mathrm{Gl}(n, \mathbb{C})$.

We will also consider compact Lie groups:

Definition 3.9. A *compact Lie group* is a topological group K that is also a compact smooth manifold, such that group operations

$$\mu\colon K \times K \to K, \quad \mu(g, h) = gh, \quad \text{and} \quad \iota\colon K \to K, \quad \iota(g) = g^{-1}$$

are smooth maps (that is, of class C^∞).

In other words, a compact Lie group K is a finite-dimensional Lie group whose underlying topological space is compact and for which the group structure is compatible with the differentiable structure. It is also a real algebraic group which embeds in some $O(n,\mathbb{R})$ (Peter-Weyl Theorem). Its points are defined by an ideal of the coordinate ring $\mathbb{R}[O(n,\mathbb{R})]$.

We call *complexification* of K and denote by $K_{\mathbb{C}}$ the complex zeros of the ideal that defines K. This will be a complex linear subgroup of $O(n,\mathbb{C})$ with coordinate ring $\mathbb{C}[G]=\mathbb{R}[K]\otimes_{\mathbb{R}}\mathbb{C}$.

There is another equivalent characterization for G to be reductive (see [BD85]).

Proposition 3.10. *A connected linear algebraic group over $\mathbb{C}$ is reductive if and only if it is the complexification of a compact Lie group.*

Example 3.11. For example, consider the unitary group $U(n)=\{M\in \mathrm{Gl}(n,\mathbb{C}) \mid M\overline{M}^{\top}=I_n\}$ which is a compact Lie group. If $M=A+\sqrt{-1}B$, we have that,

$$U(n)\cong\left\{\begin{pmatrix} A & B\\ -B & A\end{pmatrix}\in GL(2n,\mathbb{R}) \mid A^tA+B^tB=I,\ A^tB-B^tA=0\right\}$$

which in $\mathrm{Gl}(2n,\mathbb{C})$ is isomorphic to

$$\left\{\begin{pmatrix} k & 0\\ 0 & (k^{-1})^{\top}\end{pmatrix}\in GL(2n,\mathbb{C}) \mid k\in U(n)\right\}$$

If k is arbitrary in $\mathrm{Gl}(n,\mathbb{C})$, it is true that $U(n)_{\mathbb{C}}=\mathrm{Gl}(n,\mathbb{C})$. So $U(n)$ is the real locus of $\mathrm{Gl}(n,\mathbb{C})$. Similarly, $SU(n)_{\mathbb{C}}=SL(n,\mathbb{C})$.

Let K be a compact Lie group, consider the representation variety $\mathrm{Hom}(\Gamma,K)$ and the conjugation action of K on it. As K is compact, the orbits will be compact and then closed. So, it can be considered the quotient

$$\mathrm{Hom}(\Gamma,K)/K$$

which is Hausdorff. We will call it the character variety of Γ in K. It is not an algebraic variety, instead it is a *semi-algebraic* variety, that is, a finite union of sets each determined by a finite number of polynomial inequalities.

More general, if S is a real affine algebraic K-variety, then there is an equivariant closed embedding $S\hookrightarrow W$, where W is a real representation of K. Let $\mathbb{R}[W]^K$ be the ring of K-invariant in the ring $\mathbb{R}[W]$. It is known that this ring is finitely generated by polynomials $p_1,\cdots,p_d$ and the corresponding map $P=(p_1,\cdots,p_d):S\to\mathbb{R}^d$ is proper and induces a homeomorphism $S/K\cong P(S)$. There exists an ideal I such that $\mathbb{R}[S]^K\cong\mathbb{R}[p_1,\cdots,p_d]/I$. Let $Z_{\mathbb{R}}(I)$ be the real zeros of the generators of I as a subset of $\mathbb{R}^d$. Can be proven that S/K is a closed semi-algebraic subset of $Z_{\mathbb{R}}(I)$ (see [Sch89; PS85]).

Let G be the complexification of K, which will be a reductive linear algebraic group. Moreover, K will be a maximal compact subgroup of G. It can be assumed that $K\subset O(n,\mathbb{R})$ by the Peter-Weyl Theorem and then $G\subset O(n,\mathbb{C})$. Since $K\subset G$, this implies that $\mathrm{Hom}(\Gamma,K)\subset\mathrm{Hom}(\Gamma,G)$ and as G is reductive it is known that $\mathbb{C}[\mathrm{Hom}(\Gamma,G)]^G=\mathbb{C}[\mathrm{Hom}(\Gamma,G)]^K$. Consequently, the real and the imaginary parts of a set of generators for $\mathbb{C}[\mathrm{Hom}(\Gamma,G)]^G$ give a set of generators for $\mathbb{R}[\mathrm{Hom}(\Gamma,K)]^K$.

Example 3.12. Let $\Gamma=F_r$ and $K=\mathrm{SU}(n)$, then $\mathbb{R}[\mathrm{Hom}(F_r,\mathrm{SU}(n))]^{\mathrm{SU}(n)}$ is generated by the real and imaginary parts of trace functions.

Every semi-algebraic set admits a cellular decomposition, indeed

Theorem 3.13. *Let X be a closed and bounded semi-algebraic set. Then for any family $\{X_i\}$ of semi-algebraic subsets of X, there exists a cellular decomposition of X such that each X_i is a sub-complex.*

See [BCR98] on page 214 for a proof. There is an inclusion as CW complexes ([FL13] Proposition 4.5): $\iota_G : \mathrm{Hom}(\Gamma, K)/K \hookrightarrow \mathrm{Hom}(\Gamma, G)//G$.

For simplicity, we will denote $\mathrm{Hom}(\Gamma, G)//G$ (resp. $\mathrm{Hom}(\Gamma, K)/K$) by $\mathfrak{X}_\Gamma(G)$ (resp. $\mathfrak{X}_\Gamma(K)$).

In this subsection, we want to explore when it is true that $\mathfrak{X}_\Gamma(G)$ *strongly deformation retracts* onto $\mathfrak{X}_\Gamma(K)$, or $\mathfrak{X}_\Gamma(K)$ is a *strong deformation retract* of $\mathfrak{X}_\Gamma(G)$.

Definition 3.14. Let X be a topological space and $A \subset X$ a subspace. We say A is a *strong deformation retraction* of X if there is a continuous map $H : [0,1] \times X \to X$ such that

1. $H(0, x) = x$, $\forall x \in X$
2. $H(t, a) = a$, $\forall a \in A$
3. $H(1, x) \in A$, $\forall x \in X$

The map H is called a *strong deformation retraction* of X onto A. In this case, we say A is a *strong deformation retract* (SDR) of X.

A strong deformation retraction of X onto A is a homotopy between the identity of X and $j \circ r$, where r is a retraction of X to A and j is the inclusion map. If A is a strong deformation retract of X, then A and X have the same homotopy type.

We give the following definition introduced in [FL24].

Definition 3.15. The finitely generated group Γ is called *flawed* if $\iota_G : \mathfrak{X}_\Gamma(K) \hookrightarrow \mathfrak{X}_\Gamma(G)$ is an SDR, for all choices of G. It is called *flawless* if ι_G is not an SDR for any non-abelian G.

Let us give a first example.

Proposition 3.16. *If Γ is a finite group, then ι_G is a homeomorphism.*

Proof. As Γ is finite, then each $\rho \in \mathrm{Hom}(\Gamma, G)$ has finite image and so it is polystable, thus $\mathfrak{X}_\Gamma(G) = \mathrm{Hom}(\Gamma, G)/G$. Since the image is also compact, it is contained in a maximal compact subgroup of G. Choose a maximal compact subgroup K of G. All maximal compact subgroups of G are conjugate, so for every $\rho \in \mathrm{Hom}(\Gamma, G)$, there exists $g \in G$ such that $g\rho(\Gamma)g^{-1} \subset K$. By this, ι_G is surjective. Furthermore, $\mathfrak{X}_\Gamma(K)$ is compact, so ι_G is a homeomorphism. □

Example 3.17.

1. Finite groups are flawed. It is a consequence of the previous stronger fact.
2. Finitely generated free groups are flawed ([FL09]).
3. Finitely generated abelian groups, finitely generated nilpotent groups, virtually nilpotent Kähler groups are flawed ([FL14; Ber15; BF15]).

4. There are examples of finitely generated nilpotent groups Γ and non-reductive complex group G with maximal compact group K for which $\mathfrak{X}_\Gamma(G)$ is not homotopic to $\mathfrak{X}_\Gamma(K)$ ([Ber15]). This shows that the reductive assumption for G is important for the flawed notion.

5. If Γ is a hyperbolic surface group, that is, is the fundamental group of a closed orientable surface Σ of the genus greater than or equal to 2 flawless. It is known that $\mathfrak{X}_\Gamma(G)$ is homeomorphic to the moduli space of G-Higgs bundles of trivial topological type on a Riemann surface with underlying topological surface Σ (Nonabelian Hodge correspondence). The result follows from [FGN19].

6. A finite presentable group isomorphic to a free group of nilpotent groups is flawed ([FL24]). In particular, $\mathrm{PSL}(2,\mathbb{Z})$ is flawed.

Next, we describe a criterion, consequence of the Kempf-Ness Theory ([Nee85; Sch89; KN79] and Whitehead Theorem ([Hat02; Whi49], to find out if Γ is flawed.

Let V be an affine variety with rational action of G, we can construct the GIT quotient, similarly as before, $V/\!/G$. Using Lemma 1.1 of [Kem78], V can be equivariantly embedded as a closed subvariety of a finite-dimensional complex vector space $\mathbb{V}$, considering a representation $G \to \mathrm{Gl}(\mathbb{V})$.

The vector space $\mathbb{V}$ can be equipped with a K-invariant Hermitian inner product $\langle\, ,\, \rangle$ with norm $\|\ \|$. Define, for every $v \in \mathbb{V}$, the map $p_v : G \to \mathbb{R}$ by $g \to \|g \cdot v\|^2$.

Definition 3.18. A vector $X \in \mathbb{V}$ is a *minimal vector* for the action of G in $\mathbb{V}$ if

$$\|X\| \leq \|g \cdot X\|, \forall g \in G.$$

It is denoted by $\mathcal{KN}_G = \mathcal{KN}(G, \mathbb{V})$ the set of minimal vectors, called the *Kempf-Ness set* in $\mathbb{V}$ with respect to the action of G. (It depends on the choice of $\langle\, ,\rangle$ and is stable under the action of K.)

The following theorem is proved in [Sch89] using [Nee85].

Theorem 3.19. *The composition $\mathcal{KN}_G \hookrightarrow V \to V/\!/G$ is proper and induces a homeomorphism*

$$\mathcal{KN}_G/K \to V \to V/\!/G$$

where $V \to V/\!/G$ has the analytic topology.

Moreover, $\mathcal{KN}_G \hookrightarrow V$ is a K-invariant strong deformation retraction.

We apply this theorem to the case of character varieties, consider $V = \mathrm{Hom}(\Gamma, G)$ and G acts by conjugation on V. Choose r generators for Γ, this allows an embedding

$$\mathrm{Hom}(\Gamma, G) \subset G^r \subset \mathbb{V}$$

with $\mathbb{V}$ a suitable affine space where the conjugation action extends and $\mathrm{Hom}(\Gamma, G) \subset \mathbb{V}$ is a closed G-stable subvariety.

Proposition 3.20. *The Kempf-Ness set for* $\mathrm{Hom}(\Gamma, G)$ *is the closed set given by:*

$$\mathcal{KN}_G = \left\{ (g_1, \cdots, g_r) \in \mathrm{Hom}(\Gamma, G) : \sum_{i=1}^{r} g_i^* g_i = \sum_{i=1}^{r} g_i g_i^* \right\}.$$

where g_i^* is the conjugate transpose of g_i defined by a Cartan involution. Thus, we have the inclusion $\mathrm{Hom}(\Gamma, K) \subset \mathcal{KN}_G$*. The Kempf-Ness set is K-stable under conjugation.*

See the proof in [Cas+16].

The following intermediate space is needed, let $\mathfrak{Y}_\Gamma(G) := \mathrm{Hom}(\Gamma, G)/K$, it is also a finite CW complex. From 3.19,

Theorem 3.21. *$\mathfrak{X}_\Gamma(G) \cong \mathcal{KN}_G/K$ and the inclusion $\mathcal{KN}_G/K \subset \mathfrak{Y}_\Gamma(G)$ is a SDR.*

Finally, we give the criterion to decide about flawedness of Γ.

Theorem 3.22. *Let $\eta : \mathfrak{X}_\Gamma(K) \to \mathfrak{Y}_\Gamma(G)$ be the natural inclusion. The following statements are equivalent:*

1. *Γ is flawed.*
2. *η induces isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathfrak{Y}_\Gamma(G))$, for every $n \in \mathbb{N}$.*
3. *The inclusion $\mathfrak{X}_\Gamma(K) \subset \mathcal{KN}_G/K$ induces isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathcal{KN}_G/K)$, for every $n \in \mathbb{N}$.*

The ideas of the proof come from [Cas+16] and more comments in [FL24].

To determine explicitly the Kempf-Ness sets can be a very difficult task, and homotopy groups are also difficult to compute in general.

This theory can be extended to real groups and real character varieties.

A Lie group G is a *real K-reductive* Lie group if the following conditions are satisfied :

1. K is a maximal compact subgroup of G;
2. There exists a complex reductive algebraic group $\mathbf{G}$, given by the zeros of a set of polynomials with real coefficients, such that
$$\mathbf{G}(\mathbb{R})_0 \subseteq G \subseteq \mathbf{G}(\mathbb{R}),$$
where $\mathbf{G}(\mathbb{R})$ denotes the real algebraic group of $\mathbb{R}$-points of $\mathbf{G}$, and $\mathbf{G}()_0$ its identity component (in the Euclidean topology).
3. G is Zariski dense in $\mathbf{G}$.

Remark 3.23.

1. If $G \neq \mathbf{G}(\mathbb{R})$, then G is not necessarily an algebraic group (for example $G = \mathrm{Gl}(n, \mathbb{R})_0$).
2. One can think of both $\mathbf{G}$ and G as Lie groups of matrices. We will consider on them the usual Euclidean topology which is induced from (and is independent of) an embedding on some $\mathrm{Gl}(m, \mathbb{C})$.
3. $\mathbf{G}(\mathbb{R})$ is isomorphic to a closed subgroup of some $\mathrm{Gl}(n, \mathbb{R})$ (i.e., it is a linear algebraic group).
4. $\mathbf{G}(\mathbb{R})$ is a real algebraic group, hence, if it is connected, $G = \mathbf{G}(\mathbb{R})$ is algebraic and Zariski dense in $\mathbf{G}$. Condition 3. in Definition holds automatically if $\mathbf{G}(\mathbb{R})$ is connected.

Example 3.24.

1. All classical real matrix groups are in this setting.

2. G can also be any complex reductive Lie group, if we view it as a real reductive Lie group in the usual way.

3. As an example which is not under the conditions of Definition, we can consider $\widetilde{\mathrm{Sl}(n,\mathbb{R})}$, the universal covering group of $\mathrm{Sl}(n,\mathbb{R})$, which admits no faithful finite dimensional linear representation (and hence is not a matrix group).

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ the Lie algebra of $\mathbf{G}$. Fix a Cartan involution $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution

$$\theta : \mathfrak{g} \to \mathfrak{g}, \qquad \theta := \sigma\tau$$

where σ, τ are involutions of $\mathfrak{g}^{\mathbb{C}}$ that commute. The Cartan involution θ lifts to a Lie group involution $\Theta : G \to G$ whose differential is θ.

The Lie group G is embedded in some $\mathrm{Gl}(n,\mathbb{C})$ as a closed subgroup, therefore the involutions τ, σ, θ and Θ become explicit:

$$\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R}),\ \mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(n,\mathbb{C}),\ G \subset \mathrm{GL}(n,\mathbb{R}) \Rightarrow \tau(A) = -A^*$$

where $*$ denotes transpose conjugate, and $\sigma(A) = \bar{A}$. The Cartan involution θ is defined by $\theta(A) = -A^{\top}$, so that $\Theta(g) = (g^{-1})^{\top}$. The involution Θ is also the composition of two commuting involutions T and S, where T corresponds to τ and S to σ, after some modifications T can be seen as the complex conjugation composed with inverse transpose and S the complex conjugation.

We will denote by $\mathrm{Fix}(\alpha)$ the fixed points of an involution α. Thus, $\mathfrak{g} = \mathrm{Fix}(\sigma)$ and $\mathfrak{k}' := \mathrm{Fix}(\tau)$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ (so that $\mathfrak{k}'$ is the Lie algebra of a maximal compact subgroup, K', of $\mathbf{G}$). The involution σ is called a *real structure* of G.

The involution θ yields a Cartan decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}', \qquad \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{k}'$$

$\theta|_{\mathfrak{k}} = 1$ and $\theta|_{\mathfrak{p}} = -1$. The Lie algebra $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G: $K = \mathrm{Fix}(\Theta) = \{g \in G : \Theta(g) = g\}$, $K = K' \cap G$, where K' is a maximal compact subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{k}' = \mathfrak{k} \oplus i\mathfrak{p}$. The Lie algebras $\mathfrak{k}$ and $\mathfrak{p}$ are such that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

We also have a Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$: $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$ with $\theta|_{\mathfrak{k}^{\mathbb{C}}} = 1$ and $\theta|_{\mathfrak{p}^{\mathbb{C}}} = -1$.

For a finitely generated group Γ, it can be considered an inclusion

$$\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, G)$$

and so a natural map

$$\iota_G : \mathfrak{X}_\Gamma(K) \to \mathfrak{X}_\Gamma(G)$$

This map is injective by applying Remark 4.7 of [FL14] to this case which can be done by Section 3.2 of [Cas+16].

Definition 3.25. Let G be a reductive Lie group and Γ a finitely generated group.

The group Γ is said to be *strongly flawed* if there exists a K-invariant SDR from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$, for all choices of G.

The group Γ is said to be *G-flawed* if $\mathfrak{X}_\Gamma(G)$ SDR onto $\iota_G(\mathfrak{X}_\Gamma(K))$, for every maximal compact group K of G.

If Γ is G-flawed for all real reductive groups G, Γ is said to be *real flawed*.

Theorem 3.26 (Theorem 3.16 of [FL24])**.** *If Γ is real flawed then it is flawed. Conversely, if Γ is strongly flawed and the SDR commutes with a real structure on G, then Γ is real flawed.*

4 Representations and Principal bundles

4.1 Principal G-bundles

In this subsection, we remind the definition of a principal bundle on closed oriented surfaces.

Definition 4.1. Let E be smooth complex manifold, Σ be a closed oriented surface and G a Lie group. A principal G-bundle is a triple (E, Σ, π) where $\pi : E \to \Sigma$ is a smooth submersion. There is a smooth, transitive, free G-action on each fiber of π. Moreover, there exist an open cover $\{U_i\}$ of Σ and G-equivariant diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$ such that

$$(p_1 \circ \varphi_i)(\pi^{-1}(U_i)) = \pi(\pi^{-1}(U_i))$$

with $p_1 : U_i \times G \to U_i$ the projection onto the first factor.

The fiber $E_x := \pi^{-1}(x)$, for $x \in \Sigma$ is a G-torsor, choosing an element $e \in E_x$, there is a canonical identification $E_x \cong G$.

Example 4.2.

1. Consider a vector bundle $V \to \Sigma$ with rank n, the frame bundle of V is a principal $\mathrm{Gl}(n, \mathbb{C})$-bundle with fibers $E_x = \{f : \mathbb{C}^n \to V_x : f \text{ is a linear isomorphism}\}$.
2. Consider the universal cover of Σ, $\widetilde{\Sigma}$, this is a principal $\pi_1\Sigma$-bundle over Σ, in this case is a left action.

Let G be a Lie group and V a smooth manifold with a left smooth action of G on V. Consider $\pi : E \to \Sigma$ a G-bundle and the following quotient

$$E(V) := (E \times V)/G = E \times_G V$$

under the diagonal G-action, where the points of the form (y, v) and $(y \cdot g, g^{-1} \cdot v)$ are identified, for all $y \in E$, $v \in V$ and $g \in G$. We consider $\overline{\pi} : E(V) \to \Sigma$ given by $\overline{\pi}(y, v) = \pi(y)$, for every $(y, v) \in E(V)$. This is a fiber bundle. This quotient $E(V)$ is called *associated fiber bundle* to the G-bundle E.

If G is a linear algebraic group and V is a vector space with a G linear action, $E(V)$ is a vector bundle with fibers modeled on V.

Example 4.3.

1. Considering the principal $\pi_1\Sigma$-bundle over Σ, the universal cover of Σ, $\widetilde{\Sigma}$. A representation $\rho : \pi_1(\Sigma) \to G$, with G a complex algebraic group. It can be defined the following principal G-bundle:
$$E_\rho := \widetilde{\Sigma} \times_\rho G = \left(\widetilde{\Sigma} \times G\right) / \pi_1(\Sigma)$$
where $(\tilde{x}, g) \cdot \gamma = (\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} \cdot g)$, for every $(\tilde{x}, g) \in \widetilde{\Sigma} \times G$ and $\gamma \in \pi_1(\Sigma)$.
If $G = \mathrm{Gl}(n.\mathbb{C})$, E_ρ is a vector bundle over Σ of rank n.

2. Considering the adjoint representation $\mathrm{Ad}: G \to \mathrm{Gl}(\mathfrak{g})$ and a principal G-bundle E. It can be constructed the adjoint bundle

$$\mathrm{Ad}(E) := E \times_{\mathrm{Ad}} \mathfrak{g}$$

It is a vector bundle with fiber isomorphic to $\mathfrak{g}$. The equivalence relation is

$$(y, Y) \sim (y, Y) \cdot g = (y \cdot g, \mathrm{Ad}(g)^{-1}(Y))$$

for every $(y, Y) \in E \times \mathfrak{g}$ and $g \in G$.

For a principal G-bundle over Σ and V a left G-manifold. The set of sections $\Omega^0(\Sigma, E(V))$ of the fiber bundle $E(V)$ are in a bijective correspondence with maps $f : E \to V$ such that $f(y \cdot g) = g^{-1} f(y)$. Indeed, given a section $s : \Sigma \to E(V)$, a G-equivariant map $f : E \to V$ can be defined as $s(x) = (e, f(e))$, for $e \in E(V)_x$ and $x \in \Sigma$.

A *connection* on a principal G-bundle E is a $\mathfrak{g}$-valued 1-form $\omega_y : T_y E \to \mathfrak{g}$, for every $y \in E$, such that

1. $\omega_y \left(\widetilde{Y}(y) \right) = Y$, for every $Y \in \mathfrak{g}$, with $\widetilde{Y}(y) = \frac{d}{dt}\big|_{t=0} (y \exp(tY))$.
2. $R_g^* \omega_y = \mathrm{Ad}(g^{-1}) \omega_y$, for every $g \in G$.

where $R_g : E \to E$, $y \to y \cdot g$, is the right G-action on E.

An equivalent characterization is given by the differential $d\pi$ of the projection $\pi : E \to \Sigma$. For $y \in E$, the vertical tangent subspace at y is the subspace of $T_y E$ is $T_y^v E := \ker d\pi_y$. A connection is a choice of a complement to $T_y^v E$, for each $y \in E$, called horizontal tangent space and denoted by $T_y^h E$, that is, $T_y E = T_y^v E \oplus T_y^h E$. Moreover, $T_{R_g(y)}^h E = (R_g)_* T_y^h E$, for every $g \in G$ and $y \in E$.

Given a connection A in a principal bundle E, there is a covariant derivative

$$d_A : \Omega^0(\Sigma, E(V)) \to \Omega^1(\Sigma, E(V))$$

on sections of any associated vector bundle $E(V)$. Indeed, let $s \in \Omega^0(\Sigma, E(V))$ and $f : E \to V$ the G-equivariant map as above. A tensorial 1-form $\widetilde{d_A(s)}$ is defined on E by composing df with the projection $TE \to T^h E$ defined by the connection A, let $d_A(s) \in \Omega^1(\Sigma, E(V))$ be the corresponding $E(V)$-valued 1-form. The horizontal tangent spaces define a G-invariant distribution on the total space E, the obstruction for this distribution to be integrable is given by the so called *curvature*:

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(E, \mathfrak{g})$$

where the bracket $[A, A]$ is a combination of the wedge product on forms with the Lie bracket on $\mathfrak{g}$. This induces a 2-form on Σ with values in the adjoint bundle, which will be denoted equally by $F(A)$.

A connection A is said to be *flat* is $F(A) = 0$, and a principal G-bundle with a flat connection is called a *flat bundle*. This is equivalent to have a discrete structure group and by Frobenius Theorem

Proposition 4.4. *Let $E \to \Sigma$ be a flat bundle and $e \in E_x$ for some $x \in \Sigma$, Then, for any sufficiently small neighborhood U of X in Σ, there is a unique section $s \in \Omega^0(U, E_{|U})$ such that $d_A(s) = 0$ and $s(x) = e$.*

4.2 Principal bundles parametrized by representations

Let again Σ be a compact Riemann surface, with fundamental group π_1 and $\rho : \pi_1 \to G$ be a representation into a reductive group. The associated bundle construction defines a G-bundle over Σ associated to ρ. We write this principal G-bundle as $E_\rho := Y \times_\rho G = (Y \times G)/\pi_1$, where Y is a universal cover of X, and the equivalence classes are given by

$$(y,\ g) \sim \left(y \cdot \gamma,\ \rho(\gamma)^{-1} \cdot g\right), \qquad \forall \gamma \in \pi_1,\ (y, g) \in Y \times G. \tag{4.5}$$

Thus, the space of representations parametrizes holomorphic G-bundles, and we can view this construction as providing a natural map, that we call the *uniformization map*:

$$\begin{array}{rccc} \mathbf{E}: & \mathrm{Hom}(\pi_1, G)//G & \to & M_G \\ & [\rho] & \mapsto & [E_\rho] \end{array} \tag{4.6}$$

Here, M_G represents the set of isomorphism classes of G-bundles that admit a holomorphic flat connection. Note that $\mathbf{E}$ is well defined on conjugacy classes, since if ρ and σ are conjugate representations, then $E_\rho \cong E_\sigma$. Moreover, by considering the holonomy representation of a given flat G-bundle, the map $\mathbf{E}$ is easily seen to be surjective.

We will consider a special type of representations of $\pi_1 := \pi_1(\Sigma)$. Fix generators $\alpha_i,\ \beta_i$, $i = 1, \cdots, g$, of π_1 giving the usual presentation

$$\pi_1 = \left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \mid \textstyle\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \right\rangle.$$

Let G be a complex connected reductive algebraic group and denote by F_g a fixed free group of rank g, with g fixed generators $\gamma_1, \cdots, \gamma_g$. The reductive group G acts by conjugation in $\mathrm{Hom}(\pi_1, G)$ Denote by $e \in G$, the unit element of G, and consider the short exact sequence of groups,

$$1 \to \ker \varphi \hookrightarrow \pi_1 \xrightarrow{\varphi} F_g \to 1$$

where φ is the natural epimorphism given, in terms of generators, by

$$\varphi(\alpha_i) = 1, \quad \text{and } \varphi(\beta_i) = \gamma_i, \quad \forall i = 1, \cdots, g, \tag{4.7}$$

so that $\ker \varphi$ is the normal subgroup of π_1 generated by all α_i.

Definition 4.8. A representation $\rho : \pi_1 \to G$ is called a *Schottky representation* if $\rho(\ker\ \varphi) = \{e\}$, for all $i \in \{1, \cdots, g\}$.

Let $\mathcal{S}$ denote the set of *Schottky representations*, it is easy to see that

$$\mathcal{S} \cong \mathrm{Hom}(F_g, \{e\} \times G) \cong \mathrm{Hom}(F_g, G) \cong G^g \subset \mathrm{Hom}(\pi_1, G)$$

where the last isomorphism is the evaluation map: $(\sigma : F_g \to G) \mapsto (\sigma(\gamma_1), \cdots, \sigma(\gamma_g))$. Thus, $\mathcal{S}$ is a *smooth and irreducible* affine algebraic variety. The conjugation action of the reductive group G on $\mathrm{Hom}(\pi_1, G)$ restricts to an action on $\mathcal{S}$, thus it can be constructed the affine GIT quotient and also we have the homeomorphisms:

$$\mathbb{S} := \mathcal{S}//G \cong G^g//G \subset \mathbb{B} = \mathrm{Hom}(\pi_1, G)//G$$

The affine algebraic variety $\mathbb{S}$ is also *irreducible* but *singular* in general ([CFF19, Proposition 2.4]).

The notion of a good representation allows us to consider smooth points of the GIT quotient.

Definition 4.9. A representation $\rho \in \mathcal{S} \subset \mathrm{Hom}(\pi_1, G)$ is said to be *good* if ρ is good as an element of $\mathrm{Hom}(\pi_1, G)$.

Denote the set of all good (resp. good Schottky) representations by $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$ (resp. $\mathcal{S}^{\mathsf{gd}}$). Since these notions are well defined under conjugation, we can define the corresponding quotient spaces: $\mathbb{B}^{\mathsf{gd}} := \mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)//G$ and $\mathbb{S}^{\mathsf{gd}} := \mathcal{S}^{\mathsf{gd}}//G$, and, we have the inclusion $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$. The set of good representations is Zariski open in $\mathcal{S}$ (see for example [Sik12]). By [Mar00, Lemma 4.6] there exists a good representation in $\mathrm{Hom}(\pi_1, G)$, that is, $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G) \neq \varnothing$, if X has genus $g \geq 2$. The case $g = 1$ is slightly different (see Section 9 of [CFF19]).

Proposition 4.10. *[CFF19, Proposition 2.13] Let $g \geq 2$. Then, there is always a good Schottky representation $\rho : \pi_1 \to G$. Moreover, such a representation can be defined to take values in a maximal compact subgroup of G.*

Theorem 4.11. *Let $g \geq 2$. The subsets of good representations $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$ and $\mathcal{S}^{\mathsf{gd}}$ are Zariski open in $\mathrm{Hom}(\pi_1, G)$ and $\mathcal{S}$, respectively. A good representation defines a smooth point in the corresponding geometric quotient. Thus, the geometric quotients $\mathbb{B}^{\mathsf{gd}}$ and $\mathbb{S}^{\mathsf{gd}}$ are complex manifolds, and $\mathbb{S}^{\mathsf{gd}}$ is a complex submanifold of $\mathbb{B}^{\mathsf{gd}}$.*

Proof. By Proposition 4.10 there is a good Schottky representation, for $g \geq 2$. By [Sik12, Proposition 33], the subspaces of good representations in $\mathrm{Hom}(\pi_1, G)$ and $\mathcal{S}$ are Zariski open. Thus, $\mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$ and $\mathcal{S}^{\mathsf{gd}}$ are open. Since we are considering either surface groups or free groups, [Sik12, Corollary 50] shows that if $\rho \in \mathrm{Hom}^{\mathsf{gd}}(\pi_1, G)$, respectively $\rho \in \mathcal{S}^{\mathsf{gd}}$, then its class $[\rho]$ is a smooth point of $\mathbb{B}$, respectively $\mathbb{S}$. ☐

Tangent space and dimension

We begin by describing the tangent space of $\mathbb{B}$, at a good representation, in terms of the group cohomology of π_1. More generally, let Γ denote a finitely generated group and fix $\rho \in \mathrm{Hom}(\Gamma, G)$. The adjoint representation on the Lie algebra of G, $\mathfrak{g} = \mathsf{Lie}(G)$, composed with ρ, that is $_\rho : \Gamma \to G \to GL(\mathfrak{g})$, induces on $\mathfrak{g}$ a Γ-module structure, which we denote by $\mathfrak{g}_{\mathrm{Ad}_\rho}$. The following result giving an isomorphism between the Zariski tangent space of the character variety at a good representation ρ, and the first cohomology group $H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho})$, was proved by Goldman [Gol84] and Martin [Mar00].

Theorem 4.12. *For a good representation $\rho \in \mathrm{Hom}(\Gamma, G)$ we have,*

$$T_{[\rho]}(\mathrm{Hom}(\Gamma, G)//G) \cong H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}).$$

The identification between tangent spaces to character varieties and group cohomology spaces is very useful in many situations. In particular, we can use it to compute the dimension of the complex manifolds $\mathbb{B}^{\mathsf{gd}} = \mathrm{Hom}(\pi_1, G)^{\mathsf{gd}}//G$ and $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$, consisting of classes of good representations, when Γ is the fundamental group π_1 of a surface of genus g. In fact, by [Mar00, Lemma 6.2], we have, for $\rho \in \mathbb{B}^{\mathsf{gd}}$:

$$\dim Z^1(\pi_1, \mathfrak{g}_\rho) = (2g-1)\dim G + \dim Z, \quad \dim B^1(\pi_1, \mathfrak{g}_\rho) = \dim G - \dim Z,$$

and also if $[\rho] \in \mathbb{B}^{\mathsf{gd}}$, then $T_{[\rho]}\mathbb{B} \cong H^1(\pi_1, \mathfrak{g}_\rho)$ and

$$\dim T_{[\rho]}\mathbb{B} = (2g-2)\dim G + 2\dim Z \tag{4.13}$$

We now compute the dimension of $\mathbb{S}$, using the techniques of group cohomology. By the density result (Theorem 4.11), the computations can be carried out at good representations.

Proposition 4.14. *[CFF19, Proposition 7.1] Let $g \geq 2$, the dimension of $\mathbb{S}$ is given by* $\dim \mathbb{S} = (g-1)\dim G + \dim Z$.

Lagrangian submanifold

Recall that a Lagrangian submanifold $L \subset M$ of a symplectic manifold M is a half dimensional submanifold such that the symplectic form vanishes on any tangent vectors to L. It is well known that character varieties of surface group representations have a natural symplectic structure ([Gol84]), which can be constructed as follows. Consider an Ad-invariant bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$. Then, using the cup product on group cohomology

$$\cup : H^1\left(\pi_1, \mathfrak{g}_\rho\right) \otimes H^1\left(\pi_1, \mathfrak{g}_\rho\right) \to H^2\left(\pi_1, \mathfrak{g}_\rho\right), \tag{4.15}$$

and composing it with the contraction with $\langle\, , \rangle$ and with the evaluation on the fundamental 2-cycle, we obtain a non-degenerate bilinear pairing:

$$H^1\left(\pi_1, \mathfrak{g}_\rho\right) \otimes H^1\left(\pi_1, \mathfrak{g}_\rho\right) \xrightarrow{\cup} H^2\left(\pi_1, \mathfrak{g}_\rho\right) \xrightarrow{\langle, \rangle} H^2\left(\pi_1, \mathbb{C}\right) \cong \mathbb{C} \tag{4.16}$$

Under the identification of the first cohomology group $H^1\left(\pi_1, \mathfrak{g}_\rho\right)$ with the tangent space at a good representation $\rho \in \mathbb{B}^{\mathsf{gd}}$, this pairing defines a complex sympletic form on the complex manifold $\mathbb{B}^{\mathsf{gd}}$. This symplectic form is complex analytic with respect to the complex structure on $\mathbb{B}^{\mathsf{gd}}$ coming from the complex structure on G, and $\mathbb{S}^{\mathsf{gd}} \subset \mathbb{B}^{\mathsf{gd}}$ is Lagrangian.[1]

Theorem 4.17. *The good locus of the Schottky space $\mathbb{S}^{\mathsf{gd}}$ is a Lagrangian submanifold of $\mathbb{B}^{\mathsf{gd}}$.*

Proof. The restriction of the map (4.15) to $H^1\left(F_g, \mathfrak{g}_\rho\right)$ is a vanishing map:

$$\cup : H^1\left(F_g, \mathfrak{g}_\rho\right) \otimes H^1\left(F_g, \mathfrak{g}_\rho\right) \to H^2\left(F_g, \mathfrak{g}_\rho\right) = 0,$$

because free groups have vanishing higher cohomology groups (see [**B**]). Since the tangent space, at a good point, to the strict Schottky locus $\mathbb{S}$ is identified with $H^1\left(F_g, \mathfrak{g}_\rho\right)$ (see Theorem 4.12), this means that the symplectic form, defined above on $\mathbb{B}^{\mathsf{gd}}$, vanishes on any two tangent vectors to $\mathbb{S}^{\mathsf{gd}}$. Since the dimension of $\mathbb{B}^{\mathsf{gd}}$ is twice the dimension of $\mathbb{S}^{\mathsf{gd}}$ (see (4.13) and Proposition 4.14), we conclude the result. □

Relation with flat connections

Let X be a compact Riemann surface of genus g, and let M be a *compact* 3*-handlebody* of genus g with boundary $\partial M \cong X$ such that $\pi_1(M, x_0) = F_g$ and let $x_0 \in X \subset M$. Thus, the inclusion $(X, x_0) \hookrightarrow (M, x_0)$ implies the surjective map $\varphi : \pi_1 = \pi_1(X, x_0) \to \pi_1(M, x_0)$ which asigns $\alpha_i \to 1$, $\beta_i \to \gamma_i$. Let $\mathbb{F}_M(G)$ denote the moduli space of flat G-connections over M.

Theorem 4.18. *The moduli space $\mathbb{S}$, of Schottky representations with respect to φ, coincides with the moduli space $\mathbb{F}_M(G)$. That is, $\mathbb{S} = \mathrm{Hom}(F_g, G)/\!/G \cong \mathbb{F}_M(G)$.*

Proof. By hypothesis $\pi_1(M, x_0)$ is a free group of rank g, and π_1 has a "symplectic presentation" in terms of generators α_i and β_i, $i = 1, \cdots, g$, as in Equation (4.2), so that $\varphi(\alpha_i) = 1$, $\varphi(\beta_i) = \gamma_i$, $i = 1, \cdots, g$, where $\gamma_1, \cdots, \gamma_g$ form a free basis of $\pi_1(M, x_0)$. Thus, a Schottky representation $\rho : \pi_1 \to G$ with respect to φ factors through a representation of $\pi_1(M, x_0) \cong F_g$ via φ. This is

[1] For a general real Lie group, the analogous pairing defines a smooth (C^∞) symplectic structure, see [Gol84].

precisely the same as saying that the corresponding flat connection ∇_ρ on X extends, as a flat connection, to the 3-manifold M. Conversely, a flat G-connection on M induces a representation $\rho : \pi_1 \to G$ satisfying $\rho(\ker \varphi) = \{e\}$, and thus it is a Schottky representation of π_1 (with respect to φ). This correspondence is well defined up to conjugation by G, and so, we have a natural identification: $\mathbb{S} = \mathrm{Hom}(F_g, G)//G \cong \mathbb{F}_M(G)$. □

4.3 Principal Higgs bundles and branes

Let X be a compact Riemann surface of genus $g \geq 2$ and G a connected complex reductive group

Definition 4.19. A pair (E, ϕ) is a G-Higgs bundle on X: if E is a G-bundle on X and ϕ, the Higgs field, is a holomorphic section of $Ad(E) \otimes K$. ($Ad(E)$ is the adjoint bundle and K the canonical bundle of X)

Considering a notion of stability, it can be constructed $\mathcal{H}$, the moduli space of G-Higgs bundles which has a hyperkähler structure (Hitchin, [Hit87]). Denoting by (I, J, K) the choice of the three Hyperkähler complex structures we can consider submanifolds of $\mathcal{H}$ that are Lagrangian (type A) or complex (type B) with respect to each of the hyperkähler structures. Kasputin and Witten ([KW07]) called these submanifolds branes, more specifically, (B, A, A), (A, B, A) or (A, A, B)-branes. They have connection with the geometric Langlands program and mirror symmetry. In order to obtain branes we can consider anti-holomorphic involutions of X.

Anti-holomorphic involutions

Let X be a compact Riemann surface of genus $g \geq 2$ and $f : X \to X$ an *anti-holomorphic involution*. This induces an involution on $\mathbb{B}$, indeed, fixing $x_0 \in X$, f induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, f(x_0))$ and fixing γ, a path from x_0 to $f(x_0)$, the composition of the isomorphism with the conjugation by γ gives an automorphism of $\pi_1(X, x_0)$. Changing γ, the automorphism changes by composing with a inner automorphism. If we consider the good locus, $\mathbb{B}^{\mathsf{gd}}$, the involution preserves this subvariety. This can be identified with the moduli space of gauge equivalence classes of flat G-connections on X with reductive holonomy, so we get an involution of this one by pullback of connections. Now, the moduli space of gauge equivalence classes of flat G-connections on X is isomorphic to the moduli space of solutions to the Hitchin equations and this last one is isomorphic to $\mathcal{H}$. (Non-Abelian Hodge Theorem [Hit87; Sim88; Don87; Cor88]) In [BS14], it is denoted by $\mathcal{L}_G$ the set of **fixed points** of the involution in $\mathbb{B}^{\mathsf{gd}}$ (or in $\mathcal{H}$) and proved that

Proposition 4.20. *[BS14, Proposition 10] If non-empty, the set of good points of $\mathcal{L}_G$ is a smooth Lagrangian submanifold of $\mathbb{B}^{\mathsf{gd}}$.*

Following the ideas of [KW07], Baraglia and Schaposnik proved that

Theorem 4.21. *[BS14, Theorem 14] $\mathcal{L}_G$ is an (A, B, A)-brane defined on $\mathcal{H}$.*

Higgs bundles and 3-manifolds

Consider now the 3-manifold with boundary $\hat{X} := X \times [-1, 1]$, such that f defines an orientation preserving involution $\sigma : \hat{X} \to \hat{X}$ given by $\sigma(x, t) = (f(x), -t)$. The boundary of $\hat{X}$ consists of two copies of X and the boundary of the compact 3-manifold $M := \hat{X}/\sigma$, is homeomorphic to X.

Proposition 4.22. *[BS14, Proposition 43] The representations of X in G, which extend to M, belong to the (A, B, A)-brane $\mathcal{L}_G$.*

This subspace of representations can be viewed as flat G-connections on X that extend to flat G-connections over M, that is, as $\mathbb{F}_M(G)$.

4.4 Schottky representations and branes

Suppose now that we have X a compact Riemann surface with an anti-holomorphic involution $f : X \to X$, defining a real structure on X. Using the construction and notations of the subsection 4.3, let M be the compact 3-manifold whose boundary is homeomorphic to X. Then,

Theorem 4.23. *Let $f : X \to X$ be an anti-holomorphic involution such that M is a handlebody of genus g, and let $x_0 \in X \subset M$ be fixed by f. Then, the moduli space $\mathbb{S}$ of Schottky representations with respect to the map φ in (4.7) is included in the Baraglia-Schaposnik brane $\mathcal{L}_G$.*

Proof. In Proposition 4.22 it is proved the existence of an inclusion: $\mathbb{F}_M(G) \to \mathcal{L}_G \subset \mathcal{H}$. Since, by Theorem 4.18, $\mathbb{S}$ can be identified with $\mathbb{F}_M(G)$ the result follows. □

Remark 4.24. The assumption of the previous proposition is verified when the anti-holomorphic involution f has as fixed point locus the union of $g + 1$ disjoint loops and disconnected orientation double cover (see [GH81]). In this case, Proposition 4.20 says that the set of smooth points of $\mathcal{L}_G$ is a non-empty Lagrangian submanifold of $\mathcal{H}$. In a future work, we plan to further address this construction.

Under our approach, since there are good Schottky representations for every $g \geq 2$, this furnishes a proof that the set of smooth points of the Baraglia-Schaposnik brane is non-empty.

5 Exercises and Problems

1. Consider $\mathrm{Sl}(2,\mathbb{R})$ and the trace form

$$\begin{aligned} \mathrm{Tr} : \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R} &\longrightarrow \mathbb{R} \\ (v_1, v_2) &\longmapsto \mathrm{Tr}(v_1 v_2) \end{aligned}$$

   (a) Show that the trace of a matrix is invariant under conjugation, and then the trace will be Ad-invariant.

   (b) Choosing the following basis for $\mathfrak{sl}_2\mathbb{R}$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

   Prove that the trace form is given by $2x_1x_2 + y_1z_2 + z_1y_2$, which is a symmetric and non-degenerate with signature $(2,1)$.

2. Show that the group $\mathrm{SU}(p,q)$ is a real algebraic group, but is not a complex algebraic variety.

3. If G is a Lie group endowed with an analytic atlas, show that $X(\Gamma, G)$ is an analytic subvariety of G^n homeomorphic to $\mathrm{Hom}(\Gamma, G)$. And, in particular, $\mathrm{Hom}(\Gamma, G)$ has a natural structure of analytic variety and the structure does not depend on the choice of generators of Γ.

4. Consider the group $\mathrm{PSl}(2,\mathbb{R})$ which is the adjoint group of $\mathrm{Sl}(2,\mathbb{R})$.

   (a) Show that $\mathrm{PSl}(2,\mathbb{R})$ can be identified with the conjugate of the matrix group $\mathrm{SO}(2,1)^\circ$, which consists of special linear transformations of $\mathbb{R}^3$ preserving the Hermitian form $y^2 - xz$, using the following map

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

   (b) Conclude that $\mathrm{PSl}(2,\mathbb{R})$ can be identified with the group of orientation-preserving isometries of the upper half-plane $\mathbb{H}$.

   (c) Show that $\mathrm{PSl}(2,\mathbb{R})$ is homeomorphic to the unit tangent bundle of $\mathbb{H}$ (This implies that $\mathrm{PSl}(2,\mathbb{R})$ has the topology of an open solid torus).

   (d) The action of $\mathrm{PSl}(2,\mathbb{R})$ on $\mathbb{H}$ extends to its boundary $\partial\mathbb{H}$. Show that this one is isomorphic to the projective action of $\mathrm{PSl}(2,\mathbb{R})$ on $\mathbb{RP}^1$.

   (e) Consider the matrices $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose trace in absolute value is smaller than 2, prove that b and c must be nonzero. (the matrices of $\mathrm{PSl}(2,\mathbb{R})$ satisfying this are called *elliptic elements*, and its set is denoted by $\mathcal{E}$.)

   (f) Show that a matrix of $\mathrm{PSl}(2,\mathbb{R})$ is an elliptic element if and only if it has a unique fixed point in $\mathbb{H}$. In this case, if A is an elliptic element, what is its unique fixed point?

   (g) It can be defined a natural map $f : \mathcal{E} \to \mathbb{H}$. Show that f is analytic.

(h) What are the elliptic elements that fix the complex unit $i \in \mathbb{H}$? (consider the matrices

$$R_\theta := \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

, for $\theta \in (0, 2\pi)$.)

(i) Prove that every element of $\mathcal{E}$ is conjugate to a unique $R_{\theta(A)}$. Define a function $\theta : \mathcal{E} \to (0, 2\pi)$, called the *angle of rotation*, compute its expression and prove that is analytic.

(j) Consider the map $(f, \theta) : \mathcal{E} \to \mathbb{H} \times (0, 2\pi)$. Show that is an analytic diffeomorphism.

(k) The elements of $\mathrm{PSl}(2, \mathbb{R})$ whose trace absolute value is 2 are called *parabolic*. Show that an element is parabolic if and only if they have a unique fixed point of the boundary of $\mathbb{H}$. Prove that they can be characterized as belonging to one of two conjugacy classes of elements represented by $p^+ := \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (*positively parabolic*) and $p^- := \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (*negatively parabolic*).

(l) The elements of $\mathrm{PSl}(2, \mathbb{R})$ whose trace absolute value is larger than 2 are called *hyperbolic*. Show that these elements have exactly 2 fixed points in $\partial\mathbb{H}$ and they are conjugate to a diagonal matrix $h_\lambda := \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, for a unique $\lambda > 0$. (hyperbolic conjugacy classes are open annuli)

(m) Compute the centralizers of R_θ, h_λ e p^+.

5. Describe the non-regular representations $\Gamma \to \mathrm{PSl}(2, \mathbb{R})$.

6. Let $G = \mathrm{Sl}(2, \mathbb{C})$, show that the conjugation orbit, by the action of G, of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and of the 2×2 identity matrix cannot be separate.

7. Let $\rho_1 : \mathbb{Z} \to \mathrm{PSl}(2, \mathbb{R})$ be the representation given by $\rho_1(1) = p^+$ and ρ_2 be the trivial representation. Show that the orbits $\mathcal{O}_{\rho_1}$ and $\mathcal{O}_{\rho_2}$ cannot be separated by disjoint open sets in the quotient $\mathrm{Hom}(\mathbb{Z}, \mathrm{PSl}(2, \mathbb{R}))/\mathrm{Inn}(\mathbb{Z}, \mathrm{PSl}(2, \mathbb{R}))$. Is the conjugation action of $\mathrm{PSl}(2, \mathbb{R})$ in $\mathrm{Hom}(\mathbb{Z}, \mathrm{PSl}(2, \mathbb{R}))/\mathrm{Inn}(\mathbb{Z}, \mathrm{PSl}(2, \mathbb{R}))$ Hausdorff? And proper?

8. Consider a topological group G acting on a space X. Prove the following properties:

(a) $X//G$ is homeomorphic to $X/\sim$.

(b) $X//G$ satisfies the following universal property: for every $\mathcal{T}_1$ space Y, any continuous map $X \to Y$ that is constant on G-orbits factors uniquely through $\pi : X \to X//G$.

(c) There exists a natural surjective continuous map $X//G \twoheadrightarrow \mathrm{Haus}(X/G)$ such that

$$\begin{array}{ccc} X & \twoheadrightarrow & X/G \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X//G & \twoheadrightarrow & \mathrm{Haus}(X/G) \end{array}$$

commutes.

9. Compute the Kempf-Ness set for $\Gamma = F_1$ and G a linear reductive algebraic group.

10. Being flawed implies that $\pi_n(\mathfrak{X}_\Gamma(G)) \simeq \pi_n(\mathfrak{X}_\Gamma(K))$, for all n, (weakly flawed groups) since a SDR between spaces implies those spaces are homotopic and hence weakly homotopic. A problem is to find examples of weakly flawed groups that are not flawed.

11. As $\mathbb{S} \subset \mathcal{L}_G$, a problem is to study the conditions under which this inclusion is actually a bijection.

6 Project

Project Description. This project aims to explore and classify singular points in the character variety $\mathfrak{X}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$, focusing on finitely generated groups Γ, such as free groups and surface groups of low genus. We study representations up to conjugation, with particular attention to reducible and non-liftable representations, and analyze how these give rise to singularities in the moduli space.

Main Objectives:

1. Understand the structure of the character variety $\mathfrak{X}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ as a quotient space.
2. Analyze the role of reducible representations and compute their stabilizers.
3. Investigate when a representation $\bar{\rho} : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ lifts to $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{C})$, and study obstructions via group cohomology.
4. Compute Zariski tangent spaces at selected points using $H^1(\Gamma, \mathfrak{sl}_2^{\mathrm{Ad}\circ\bar{\rho}})$, and detect singularities.
5. Carry out a case study for $\Gamma = F_2$, including explicit examples of singularities.

Expected Outcomes:

- Classification of some singular points in low-dimensional $\mathrm{PSL}(2,\mathbb{C})$ character varieties.
- Description of the effect of non-liftability on the local structure of the character variety.
- Concrete examples of singularities arising from reducible representations.

Selected References:

- [Gol84]
- [Sik12]
- [FL13]

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